Math 256: Lab #2 (modified Lab 2.3 from textbook) Due: Wednesday, October 25.

Autonomous second-order differential equations are studied numerically by reducing them to first-order systems with two dependent variables. In this lab you will use the computer and analytic techniques to analyze three somewhat related second-order equations. In particular, you will analyze phase planes and $y(t)$- and $v(t)$-graphs to describe the long-term behavior of the solutions.

In sections 2.1 and 2.3, we discuss the most classic of all second-order equations, the harmonic oscillator. The harmonic oscillator is

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

It is an example of a second-order, homogeneous, linear equation with constant coefficients. In the text we explain how this equation is used to model the motion of a spring. The force is assumed to obey Hooke's Law (the force is proportional to the amount the spring is compressed or stretched). The force due to damping is assumed to be proportional to the velocity. In your lab report you should describe the motion of the spring assuming the values of m, b, and k in the charts below.

Your report should discuss the following:

1. Undamped Harmonic oscillator. The first equation we will study is the harmonic oscillator with no damping (b=0). For each choice of the m and k values in the chart below find (i), (ii), and (iii) and answer the follow up questions 1a), 1b) and 1c).

	mass, m	spring constant, k
Choice 1	20	5
Choice 2	5	20

(i) Write the system of differential equations which correspond to the harmonic oscillator with coefficients m and k.

(ii) Find the general solution to the undamped harmonic oscillator equation, i.e. find $y(t)$ and $v(t)$. Solve the two initial value problems for $(y_0, v_0) = (2, 0)$ and $(y_0, v_0) = (0, -4)$.

(iii) Examine the solutions using both the phase plane and graphs of the solutions as a function of time. Sketch or include a printout of the phase space, and draw the graphs of $y(t)$ and $v(t)$ for the initial value problems $(y_0, v_0) = (2, 0)$ and $(y_0, v_0) = (0, -4)$.

Answer the following questions:

1a) Are the solutions periodic? If so, what does the period appear to be? Include and label any periodic behavior in your graphs of $y(t)$ and $v(t)$ for the two different sets of initial conditions.

1b) For each of the two sets of initial conditions, what are the physical interpretations of your initial conditions? How are these initial conditions reflected in your graphs in the phase plane and the $y(t)$- & $v(t)$-graphs?

1c) What can you say globally about the behavior of the undamped harmonic oscillator over time? Make a conjecture about the behavior of solutions for other values of m and k. Compare and contrast the solutions for harmonic oscillator using the variable in Choice 1 versus Choice 2. How do the phase planes differ? Are there values of m and k for which the solutions spiral counter-clockwise about the origin?

2. Harmonic Oscillator with damping For each choice of the m, k, and b values in the chart below repeat 1.(i)-(iii)

	mass, m	spring constant, k	damping constant, b
Choice 1	20	5	2
Choice 2	5	20	2
Choice 3	20	5	20
Choice 4	5	20	20

Answer the following questions:

2a) Of the four choices above, which choices of m, k, and b have periodic solutions? Which have straight line solutions?

2b) What general conditions on m, k and b guarantee the system has straight line solutions? (prove your assertion)

2c) What general condition on m, k, and b guarantee the system has spiral solutions? (prove your assertion) What general conditions on m, k and b guarantee the origin is a spiral source? (prove your assertion) What general conditions on m, k, and b guarantee the origin is a spiral sink? (prove your assertion) If the eigenvalues are complex, is there a condition on b and k that gives solutions which spiral counter-clockwise around the origin? (explain)

2d) Are there any values of m, k, and b for which the origin is a source or saddle equilibrium? If so, what is the condition on m, k, and b. If not, explain why not.

3. Harmonic oscillator with nonlinear damping For this question we will use numerical and qualitative techniques to analyze phase planes and determine the long-term behavior of the solutions to the differential equation

$$m\frac{d^2y}{dt^2} + b\left|\frac{dy}{dt}\right|\frac{dy}{dt} + ky = 0,$$

using the m, k, and b values in the chart below.

	mass, m	spring constant, k	damping constant, b
Choice 1	20	5	2
Choice 2	5	20	2
Choice 3	20	5	20
Choice 4	5	20	20

Note that even with the same value of the parameter b the damping force in this equation and the equation in problem 2 have the same magnitude only for velocity ± 1. Also, notice that the sign of the term $\left|\frac{dy}{dt}\right|\frac{dy}{dt}$ is the same as $\frac{dy}{dt}$, hence the damping force is always directed opposite the direction of motion. The difference between this equation and the classical harmonic oscillator with damping is the size of the damping for large and small velocities. One of the many examples for which this is a better model than linear damping is the drag on an airplane tires from wet snow or slush. Drag from only four inches of slush was enough to cause the 1958 crash during take-off of plane carrying the Manchester United soccer team. Currently, large airplanes are allowed to take off and land in no more than a 1/2 inch of slush.*

(i)' Examine solutions and their graphs in the phase plane using Euler's method for systems. Are the solutions periodic? If so, what does the period seem to be? Although we have not yet discussed the Runge Kutta Method for systems, use this option in the HPGSystemSolver and compare your findings to those using Euler's Method.

(ii)' Describe (in words and in a graph) the solutions to three different initial value problems $(y_0, v_0) = (1, 3)$, $(y_0, v_0) = (3, -2)$, and $(y_0, v_0) = (-0.1, -0.1)$. How to different values of m, k, and b impact the long-term behavior of the solution? What is the long-term behavior of the system? Is the long-term behavior of the solutions different for different initial conditions? If so, give a description of the corresponding initial conditions.

4. Nonlinear second-order equation Finally, consider a somewhat related second-order equation where the damping coefficient b is replaced by the factor $(y^2 - \alpha)$; that is,

$$m\frac{d^2y}{dt^2} + (y^2 - \alpha)\frac{dy}{dt} + ky = 0.$$

Is it reasonable to interpret this factor as some type of damping? Provide a complete description of the long-term behavior of the solutions for the values of m, k, and α in the chart below. Are the solutions periodic? If so, what does the period seem to be? Explain why this model is not a good model for the mass-spring system. Give an example for some other type of physical or biological phenomenon that could be modeled by this equation.

	mass, m	spring constant, k	α
Choice 1	20	5	2
Choice 2	5	20	3
Choice 3	5	40	10

*See Stanley Stewart, *Air Disasters*, Barnes & Noble, 1986.