Integrated Calculus & Physics
Day-by-day Learning Outcomes & Lecture Notes

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December 2, 2011

Contents

Chapter 1

Vectors

1.1 Vectors, graphically and algebraically, from HH 13.1 and 13.2

Outline of topics: **(60+ minutes)**

- defn of vector, defn of scalar,
- geometric understanding of vectors, displacement vectors, vector sum, mult vector by scalar, vector subtraction
- parallel vectors
- $\vec{\imath}, \vec{\jmath}$ standard basis vectors, unit vectors
- find a vector from P to Q
- length of a vector
- vector sum & scalar mult in coordinates
- finding coordinates of a geometric vector
- **Homework: Section 13.1 2, 4, 14, 16 (exercises); 28, 30, 32, 33, 37, 41 (problems)**

Learning Outcomes:

- Distinguish between vector and scalar quantities
- Articulate examples of applications of vectors
- Draw: vectors, vector sum, mult by scalar, vector subtraction
- Determine whether two vectors are parallel
- List and draw standard basis vectors
- Given a vector graphic (picture), determine the vector in coordinates (notation) and vice versa.

$\boxed{\textbf{Lecture 1.}}$

Definition 1.1.1. A **vector quantity** is a quantity that requires a size and a direction to describe precisely.

Example 1.1.2. These are examples of vector quantities.

- My house is 4 miles North and 3 miles East of here.
- The stock for company ABC rose 12 points today.
- The tornado is moving 30 mph northeast from Kansas City.

Definition 1.1.3. A **scalar quantity** is a quantity that requires only one number to describe precisely.

Example 1.1.4. These are examples of scalar quantities.

- height, length, volume, mass, temp
- speed of 78 mph (this is enough to know you are speeding!)

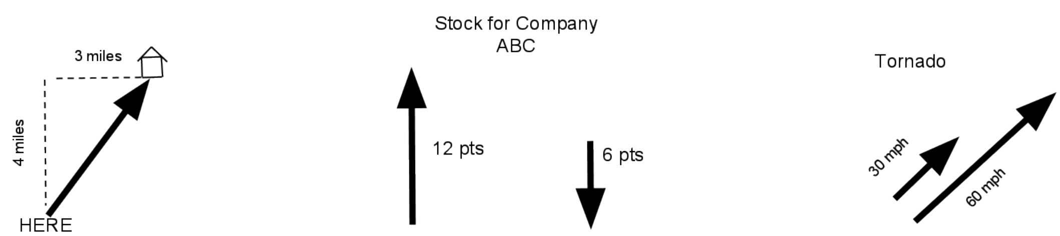


Figure 1.1: Vector examples drawn geometrically.

Example 1.1.5. Displacement vectors. A robot travels 10 miles NE and then 40 miles SE. Draw the displacement vectors for each portion of the robot's travels and draw the total displacement vector for the total of the two trips.

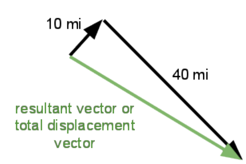


Figure 1.2: Vectors describing the two trips the robot makes and the vector for the total trip.

This motivates the definition of the sum of two vectors. Figure 1.3 explains how to find the sum of two vectors graphically, and Figure 1.4 shows that $\vec{a} + \vec{b}$ is equal to the vector $\vec{b} + \vec{a}$. Hence vector addition is commutative.

Example 1.1.6. Another robot travels 3 times as far as the previous robot in the NE direction and the same amount in the SE direction. Draw the total displacement vector for this second robot.

Example 1.1.7. Vector subtraction. $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$ (Draw pictures in class.)

Definition 1.1.8. Two vectors $\vec{a}$ and $\vec{b}$ are called **parallel** if $\vec{a} = \lambda\vec{b}$, for some scalar λ

Example 1.1.9. Draw pictures of an arbitrary vector $\vec{a}$ and $-\frac{1}{2}\vec{a}$ and $5\vec{a}$, as examples of parallel vectors.

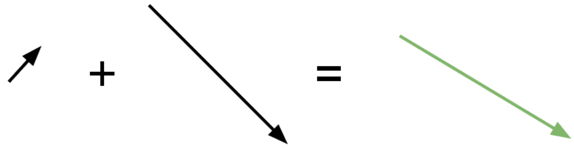


Figure 1.3: To graphically sum two vectors $\vec{a}$ and $\vec{b}$: Place the tail of $\vec{b}$ at the tip of $\vec{a}$, then draw a new vector by connecting the tail of $\vec{a}$ with the tip of $\vec{b}$. This new vector is called $\vec{a}+\vec{b}$.

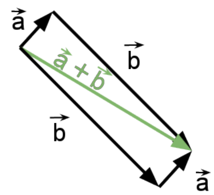


Figure 1.4: The same total displacement vector is found by summing $\vec{a}+\vec{b}$ and $\vec{b}+\vec{a}$.

Definition 1.1.10. The vectors $\vec{\imath}$ and $\vec{\jmath}$ are unit vectors in the positive x and y directions respectively. A vector is a **unit vector** provided that it has length one. (Draw xy-plane and the vectors $\vec{\imath}$ and $\vec{\jmath}$.)

Example 1.1.11. Draw the vector $\vec{w} = 3.5\vec{\imath} + -6\vec{\jmath}$. Explain component vectors. Briefly mention vectors in 3$-$space.

Example 1.1.12. Find the vector from the point $P_1 = (6, 2, 1)$ to the point $P_2 = (-1, 4, 2)$. (Or scale this back to 2d version. Emphasize the usefulness of components.)

Definition 1.1.13. The length of a vector $\vec{v} = a\vec{\imath}+b\vec{\jmath}$, denoted $||\vec{v}||$, is calculated using the Pythagorean Theorem $||\vec{v}||^2 = ||a\vec{\imath}||^2 + ||b\vec{\jmath}||$. Thus

$$||\vec{v}|| = \sqrt{a^2 + b^2}, \text{ for } \vec{v} = a\vec{\imath} + b\vec{\jmath}.$$

Example 1.1.14. Calculate the length of $6\vec{\imath} - 2\vec{\jmath}$. Calculate the length of $\lambda\vec{v}$.

Example 1.1.15. Given the geometric representation of the vector $\vec{v}$ find the component vectors. ($\vec{v}$ is 5 units long and make an angle of $\frac{\pi}{6}$ with the positive x-axis.)

Example 1.1.16. Find a unit vector in the direction of $\vec{v} = \langle 2, -1 \rangle$.

ConcepTests

Possible ConcepTests or Quiz Questions:
1. Consider the vector $\vec{v} = -\vec{\imath} + 5\vec{\jmath}$. Find a vector that
(a) Is parallel but not equal to $\vec{v}$.
(b) Points in the opposite direction of $\vec{v}$.
(c) Has unit length and is parallel to $\vec{v}$.

ANSWER: (a) $3\vec{v}$, (b) $-\vec{v}$, (c) $\vec{v}/||\vec{v}||$ Other answers are possible. Follow-up Question. How many vectors can you find in each case?

2. Decide if each of the following statements is true or false:
(a) The length of the sum of two vectors is always strictly larger than the sum of the lengths of the two vectors.
(b) $||\vec{v}|| = |v1| + |v2| + |v3|$,where $\vec{v} = v1\vec{\imath} + v2\vec{\jmath} + v3\vec{k}$.
(c) $\pm\vec{\imath}, \pm\vec{\jmath}, \pm\vec{k}$ are the only unit vectors.
(d) $\vec{v}$ and $\vec{w}$ are parallel if $\vec{v} = \lambda\vec{w}$ for some scalar λ.
(e) Any two parallel vectors point in the same direction.
(f) Any two points determine a unique displacement vector.
(g) $2\vec{v}$ has twice the magnitude as $\vec{v}$.

ANSWER: (a) False (b) False (c) False (d) True (e) False (f) False (g) True
Follow-up Question. Find counterexamples for the false statements.

End of Hour 1

Michaela: motion diagrams, velocity, ... **(Michaela's time:)**

Chapter 2

Calc I Review

2.1 Graphs

Learning Outcomes:

- Determine a list of functions that could be used to model given data.
- Draw the graph of the absolute value of a function
- Articulate differences between linear, power and exponential functions (achieve this perhaps in a lab style problem set. Graph $y = 2x$, $y = x^2$, $y = 2^x$. Write about observed differences. Given several data sets determine whether they are best modeled via linear, power or exponential functions.)

2.2 Trig Review

Learning Outcomes:

- Apply the definition of the trig functions to produce the Pythagorean Identity
- Create model from given data. (for the tides of Oregon Bay using 12.4 hours between high tides and a 15 meter difference between high and low tide and the first high tide of the day at T hours.

2.3 Limits

Learning Outcomes:

- Articulate the defn of a limit as a “process"
- Articulate pitfalls of attempting to evaluate limit by simply plugging in the value
- Evaluate limits from graph, algebraically, from table and make judgements regarding accuracy

Lecture 2.

Review Graphs via Handout. (perhaps email this to students to bring to the first day of class) **(10 minutes)**

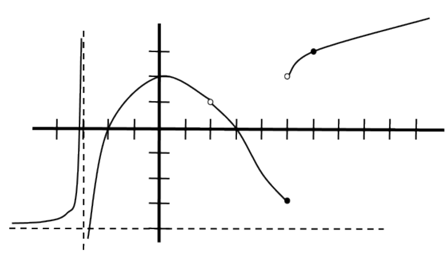
- Fill in the blank handout on graphs. Know the basic shape of these graphs without a calculator.
  - lines: $y = mx + b$, $y - y_1 = m(x - x_1)$
  - parabolas: $y = ax^2 + bx + c$, $y = a(x - h)^2 + k$
  - circles: $(x - h)^2 + (y - k)^2 = r^2$, circle of radius r with center (h, k)
  - absolute value: $y = |x|$
  - root functions: $y = \sqrt{x}$, $y = \sqrt[3]{x}$
  - exponential function: $y = 2^x$, $y = e^x$, $y = 3^x$, $y = 2^{-x}$
  - log functions: $y = \log_{10}(x)$, $y = \ln(x)$
  - trig functions: $\sin(x), \cos(x), \tan(x)$
  - graph shifting, stretching & reflecting
- find an equation that matches the data
- find an equation that matches the words (do one like this in class)
- **Homework:** WeBWorK (WW) and in-class quick quiz (hold up arms, what function am I?).

Review Trig. **(20 minutes)**

- Unit circle defn & SOH, CAH, TOA. *why do these both give the same values?*
- definition of sine and cosine and values at $0, \pi/6, \pi/4, \pi/3, \pi/2$
- *Why do the graphs of sine and cosine look the way they do? why period 2π? why amplitude 1?*
- pythagorean identity. *why is* $\sin^2(x) + \cos^2(x) = 1$*?*
- **Homework:** WW

Review of Limits. Definition and how to calculate. **(30 minutes)**

- *What is a limit? Can we always just plug in the value?*
- *When can we plug in the value?* recall definition of continuity.
- Calculate limits from the graph.



- Calculate limit of polynomial algebraically
- Make a table of values to calculate $\lim\limits_{h \to 0} \dfrac{\sin(h)}{h}$.
- *How reliable is calculating a limit from a table of values?*

- Use table of values to calculate $\lim_{h \to 0} \sin\left(\frac{\pi}{h}\right)$ for $h = 0.1, 0.01, 0.001, 0.000234$
- **Homework:** WW

ConcepTests

Examples of good conceptests are printed out in notes.

End of Hour 2

2.4 The Derivative

The most important limit! **(50 minutes)**

- definition of derivative, slope of tangent line, algebraic example
- example $y = V(t)$, volume as a function of time. What does $V'(t)$ mean? Units. Graph of $V'(t)$ from graph of $V(t)$.
- Inga: cubic position function $y = s(t)$, determine the times at which the object turns around
- Graph $f'(x)$ from discontinuous, not always differentiable graph of $f(x)$. (recall definition of continuity)
- Approximate derivative from a table (approximate instantaneous velocity using average velocity)
- The derivative and the shape of the curve (increasing/decreasing, concave up/down).
- **Homework: part WW and some from HH textbook (pick these out!!).**

Learning Outcomes:

- Use the definition of the derivative to calculate derivative functions
- Recognize the derivative as slope of tangent line and rate of change
- Approximate derivative from data and make judgements about accuracy
- Use information about the derivative function, f' to draw conclusions about the shape of the original curve f

End of Hour 3

2.5 Functions, Models, Optimization, Rates of Change, Derivative Rules

Outline of topics:

- Optimization (Illumination example - review quotient and chain rules)
- Modeling with rates of change (water entering a tank - when will it be full?)

- Review of differentiation rules (drill problems in HW)

Learning Outcomes:

- Optimize using derivative (review quotient rule and chain rule).
- Recognize rate of change as a derivative (wording?)
- Problem solve using rates of change (wording?)
- **Homework: Section 4.4, problems 3, 10, 33**
- **Homework: Section 4.6 problems 30, 42**
- Derivative rules and formulas for polynomials, piecewise functions, trig functions, exponential functions, log functions, & inverse trig **(handout)**
- **Homework: WW**

Lecture 1.

Example 2.5.1. A light L is suspended at a height h meters above the floor. Suppose P is a point on the floor 10 meters away from the point directly below the light O . Let θ denote the angle $\angle OLP$

The **illumination** at P is inversely proportional to the square of the distance from the point P to the light L and directly proportional to the cosine of the angle θ. How far from the floor should the light be placed to maximize the illumination at P. **(20 minutes).**

Solution: First we draw a picture. Then use the definition of proportional to get the formula $I = \frac{k\cos(\theta)}{r^2}$. Next translate into a function of h for $h \geq 0$,

$$I = \frac{kh}{(h^2+10^2)^{3/2}}.$$

Example 2.5.2. Water is pouring into a cylindrical tank (swimming pool). When will the tank be full?

Is it pouring in at a constant rate? (suppose so) Is the tank initially empty? (suppose so) What if we know that when the depth is 4 feet, the depth is increasing at a rate of 0.2 ft/sec. How fast is the volume of the tank changing? What if the height of the cylinder is H feet and the radius is R feet. When will the tank be full (in seconds, in minutes, in hours)? **(20 minutes)**

What if it isn't pouring in at a constant rate? (hmmm.... this is a harder problem, but we can figure it out with calculus. More on this later.)

2.6 L'Hopital's Rule

Outline of topics:

- What is $\frac{0}{0}$? or $\frac{\infty}{\infty}$? Give examples showing they could equal anything.
- Statement of L'Hopital's Rule, examples: a few $\frac{poly}{poly}$, $\frac{\sin(x)}{x}$, $x \cdot e^{-x}$ **(20 minutes)**

Learning Outcomes:

- Articulate the characteristics of an indeterminate form
- Identify indeterminate forms
- Apply L'Hopital's Rule to evaluate limits of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$
- **Homework: WW and the problem below.**

Part a. Evaluate $\lim_{x\to\infty} x\ln(1+\dfrac{1}{x})$

Part b. Use your calculation above to evaluate

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left((1+\frac{1}{x})^x\right)}$$

and explain why the two limits above are equal.

Part c. Read the website

http://math.ucsd.edu/∼ wgarner/math4c/textbook/chapter4/compoundinterest.htm

and give a brief explanation *in your own words* of **what, why, and how** the limit in Part b is related to Compound interest. Here, "brief" means about half a page hand-written.

End of Hour 4

Michaela: going back to motion diagrams, problem solving strategy using motion diagrams, position, velocity and acceleration graphs **(?? hours)**

2.7 The Integral and FTC

Outline of topics:

- Measuring distance traveled from velocity.
- Relationship between distance traveled and area under curve. What if the velocity is sometimes negative?
- The definite integral (defn, numerical calculation, graphical calculation)
- The FUNdamental Theorem of Calculus & **The Net Change Theorem. Very IMPT!!**
- Change in position from velocity, difference between change in position and total distance traveled
- Finding antiderivatives graphically
- Finding antiderivatives analytically
- u-substitution **(100 minutes)**
- **Homework: Section 5.1 problems 8, 16, 26; Section 5.2 problem 10; Section 5.3 problems 1, 2, 14, 38; Section 6.1 problem 13, 16; Section 6.2 and u-sub WW**

End of Hour 6, End of Calculus I Review

(Times given in black indicate Inga's approx lecture time. I have 7 hours scheduled for this material and I might need it.)

Michaela: uniform motion, equations of motion, free fall, motion on an inclined plane, 2d-stuff, projectile motion, circular motion **(3+ hours)**

Total time thus far: ?? hours

Chapter 3

Parametric and Polar Equations

3.1 Parametric Equations, mostly from HH Section17.1

Outline of topics:

- Purpose, uses and defn of parametric equations.
- Building intuition about parametric equations, Parametric equations of a line, circle, more.
- Two particles move through space according to given parametric equations, do they collide?
- Parametrized curves as vector-valued functions

- Purpose, uses and defn of parametric equations.

  Parametric equations are used to describe curves in the plane that are not functions and they can also be used to describe an object moving in the plane or in space.

  As an example of a curve that is not a function, consider the circle $x^2 + y^2 = r^2$. If you tried to describe this curve via a function you'd end up with two equations $y = \pm\sqrt{r^2 - x^2}$, the positive equation describes the top half of the circle and the negative describes the bottom half. There is no one function that describes the whole circle and these two functions are quite unpleasant to work with. *Can we integrate this easily? Nope!*

  A parametric equation can be used to describe curves such as this by introducing a new variable, t, and instead of trying to find a formula $y = f(x)$ that describes the curve we will be looking for two functions $x = p(t)$ and $y = q(t)$ describing the x and y coordinates of the curve respectively.

  Definition 3.1.1. A parametric equation is a set of equations that describes a curve in the plane or 3-space in terms of parameter t. There are various notations for parametric curves:

  $(x, y) = (p(t), q(t))$ and $\left\{ \begin{array}{lcl} x & = & p(t) \\ y & = & q(t) \end{array} \right.$ describe a curve in the plane

  $(x, y, z) = (p(t), q(t), r(t))$ describes a curve in 3-space in terms of the parameter t.

  Let's start by looking at parametric equations for curves that *are* functions before moving on to curves that cannot be described by a function.

- Building intuition about parametric equations, Parametric equations of a line, circle, more.

  Example 3.1.2. Consider the parametric equation $(x, y) = (3t + 6, t - 1)$. Let's plot points to determine the graph... This equation describes a line of slope $\frac{1}{3}$ with y-intercept $y = -3$. Let's **generalize** this to observe something more general. What curve does the parametric equation $(x, y) = (at + b, ct + d)$, where a, b, c, d are constants, describe? Answer: a line of slope $\frac{c}{a}$ with y-intercept ... ? (A problem for you to think about more at home.)

Great, now we know how to recognize a family of parametric equations as lines. How about the other way around? Given a line can we find a parametric equation that describes it? We look at this question in the next example.

Example 3.1.3. Next suppose we are given information about an object that is moving along a straight line in the plane. This object is at the position $(2, 0)$ at time $t = 0$ and at $(0, 6)$ at time $t = 1$. How can we find a parametric equation describing the motion of this object in the plane? Note: this problem isn't completely well posed, but assuming the object is moving at constant speed we can use the parametric equations from the previous example.

$$\boxed{(x, y) = (at + b, ct + d) \qquad \text{parametric equation of a line}}$$

We are given that $(a \cdot 0 + b, c \cdot 0 + d) = (2, 0)$ and $(a \cdot 1 + b, c \cdot 1 + d) = (0, 6)$. Thus solving for a, b, c and d, we see the parametric equation $(x, y) = (-2t + 2, 6t)$ describes the motion of the object.

Now let's move on to describing the unit circle, $x^2 + y^2 = 1$, parametrically. Any parametric equation $(x, y) = (p(t), q(t))$ that describes a circle must satisfy the equation of a circle. That means we want to find functions $p(t)$ and $q(t)$ that satisfy

$$(p(t))^2 + (q(t))^2 = 1.$$

Do we know of any nice functions that would satisfy this equations? Yes! The sine and cosine functions. Of course the sine and cosine functions are defined parametrically in terms of the angle θ.

Example 3.1.4. Graph the parametric equations given below and write several sentences that explain a generalization of your findings.

1. $(x, y) = (\cos(t), \sin(t))$
2. $(x, y) = (\cos(2t), \sin(2t))$
3. $(x, y) = (2\cos(t), 2\sin(t))$
4. $(x, y) = (-3\sin(\pi t), 3\cos(\pi t))$
5. $(x, y) = (7 + \cos(t), -1 + \sin(t))$

What can you say about curves of the form $(x, y) = (a + b\cos(ct), k + l\sin(mt))$ or $(x, y) = (d + e\cos(ft), g + h\sin(jt))$?

Handout a worksheet for this example. Students can get started in-class and finish at home.

Example 3.1.5. Graph the parametric equation $(x, y) = (3t^2, 4t - 1)$... a parabolic curve. Show how to eliminate the t variable.

Example 3.1.6. Graph the parametric equation $(x, y) = (3t^2, 4t^2 - 1)$... a line? no a ray! The object turns around!! Show how to eliminate the t variable; however, here the t^2 influences the graph by forcing $x \geq 0$ and $y \geq -1$. Thus the graph is a ray, not a line.

- Two particles move through space according to given parametric equations, do they collide?

Example 3.1.7. Suppose the motion of two objects in the plane are described parametrically by the equations below.

$(x, y) = (t + 1, t - 1)$ $\qquad$ $(x, y) = (t^2 - 1, 3t + 3)$

Do the objects collide? We can graph the line and parabola equations above to see that the curves have two points of intersection. But are points of intersection the same as collision points? NO. To have a collision there must exist a time t at which the both objects are at the same point. Thus we are looking for a time t such that *both* of the equations below are satisfied.

same x-coordinate: $t + 1 = t^2 - 1$ $\qquad$ same y-coordinate: $t - 1 = 3t + 3$

The first equation has solutions $t = -1, 2$ and the second has solution $t = -2$. There is no t that satisfies both equations, so there is no collision point.

- Parametrized curves as vector-valued functions

  A point in the plane (x, y) can also be described by a position vector $\vec{r} = x\vec{i} + y\vec{j}$. We can write the parametric equations $(x, y) = (p(t), q(t))$ as $\vec{r}(t) = p(t)\vec{i} + q(t)\vec{j}$.

  For example, circular motion in the plane can be written as $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$. Using this notation, the vector-valued function below describes what familiar curve in the plane?

  $$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \qquad \text{for constant vectors } \vec{r}_0, \vec{v}$$

- **Homework: Section 17.1 problems 1-4, 10, 15, 18, 28, 42, 57, 59, 70, WW**

3.2 Derivatives of Parametric Equations, mostly from HH Section 17.2

Outline of topics:

- dy/dx formula for parametric curve, the slope of the curve in the plane (this quantity does NOT depend on the parametrization!)
- Vector quantities: velocity vector, acceleration vector, force vector
- Uniform Circular motion
- Motion along a line
- Arc length (before Arc Length of a function!?)

Lecture 1.

- dy/dx formula for parametric curve, the slope of the curve in the plane (this quantity does NOT depend on the parametrization!)

  Given a parametric curve $(x, y) = (p(t), q(t))$ that is a function for some intervals of t-values (see picture), we can use the Chain Rule to find a formula for $\frac{dy}{dx}$, the slope of the line tangent to the parametrized curve.

  Given the parametrized curve $(x, y) = (p(t), q(t))$. We can easily differentiate with respect to t to get $\frac{dx}{dt} = p'(t)$ and $\frac{dy}{dt} = q'(t)$. But how are these quantities related to the quantity $\frac{dy}{dx}$ that describes the slope of the line tangent to the plane curve? Working on an interval of t-values for which y is a function of x and x is a function of t and applying the Chain Rule, we have

  $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Solving for $\frac{dy}{dx}$ we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Now we can use this formula for the slope of the tangent line to help us sketch parametric curves in ways analogous to how $f'(x)$ can be used to sketch the graph of $f(x)$.

Example 3.2.1. Use the formula for $\frac{dy}{dx}$ to sketch the graph of the parametric curve $(x, y) = (t^3 - 3t, t^2 + 4)$.

First notice that from the end behavior of cubic and quadratic functions the x-coordinate of this curve will get infinitely large in both the positive and negative directions, however the y-coordinate of the curve will never attain values below 4.

We calculate $\frac{dy}{dx} = \frac{2t}{3t^2-3}$. This implies that the curve has a horizontal tangent line, i.e. zero slope tangent line, at $t = 0$, and the tangent line is undefined at $t = \pm 1$. Finding the points (x, y) on the curve that correspond to these t-values we have

$$\text{a horizontal tangent line @ } t = 0 \text{ and } (0, 4)$$

$$\text{a vertical tangent line @ } t = 1 \text{ and } (-2, 5)$$

$$\text{a vertical tangent line @ } t = -1 \text{ and } (2, 5)$$

Analyzing the positivity and negativity of the slope of the tangent line on the t-value intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ we have

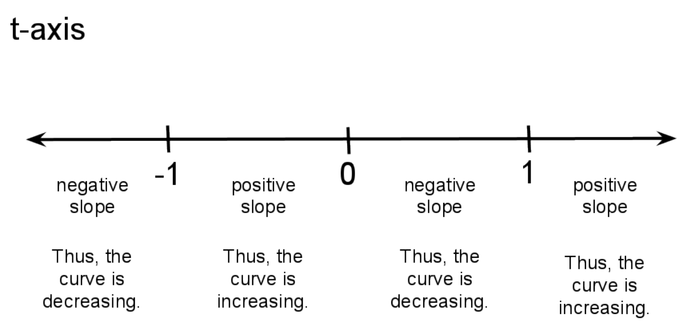


Figure 3.1: Intervals of t-values over which the curve is increasing/decreasing.

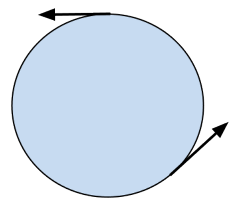
Thus the curve can be sketched as ...

- Vector quantities: velocity vector, acceleration vector, force vector

Definition 3.2.2. The **velocity vector**, $\vec{v}$ of a moving object is a vector such that:
• the magnitude of $\vec{v}$ is the speed of the object
• the direction of $\vec{v}$ is tangent to the object's path in the direction of motion.

Example 3.2.3. Drawing velocity vectors.

A ball attached to a rope is being swung in a circular path (bring in model and swing it). The length of the rope is 0.5 meters and the ball makes 2 revolutions per second. Find the speed of the ball and draw the velocity vector at two different times.



The ball moves at a constant speed around the circle of radius $r = 0.5$ meters, completing 2 revolutions per second. The length of one revolution is the circumference of the circle $C = 2\pi r = \pi$ meters. Thus the speed of the ball is $\frac{\Delta\text{position}}{\Delta\text{time}} = \frac{2\pi\text{meters}}{1\text{second}} = 2\pi$ meters per second. Hence the magnitude of the velocity vector is 2π meters per second. The direction of the velocity vector is tangent to the circle in the direction that the ball would travel if the rope was cut. (See figure - circle of radius 1/2 with two tangent vectors of length 2π.)

Definition 3.2.4. Calculating velocity vectors.

Given a vector-valued position function, $\vec{r}(t)$, we calculate the velocity vector, as in the one-dimensional case, by taking a limit. The quantity $\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$ is the displacement vector between the positions at times t and $t + \Delta t$. So the average velocity of the object over this interval is

$$\text{average velocity } = \frac{\Delta\vec{r}}{\Delta t}.$$

Taking the limit as $\Delta t \to 0$, we get the instantaneous velocity vector at time t:

The velocity vector, $\vec{v}(t)$, of a moving object with position vector $\vec{r}(t)$ at time t is given by

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

- Uniform Circular motion

Example 3.2.5. Components of the velocity vector.

From the previous example, a ball attached to a rope is being swung in a circular path (bring in model and swing it). The length of the rope is 0.5 meters and the ball makes 2 revolutions per second.

Find a vector equation for the position of the ball at time t assuming the position of the ball in its plane of movement is $(x, y) = (0.5, 0)$ at time $t = 0$.

The model for this motion is $\vec{r}(t) = 0.5\cos(\omega t)\vec{i} + 0.5\sin(\omega t)\vec{j}$, where ω chosen to reflect the period of 1 revolution every half second. The cosine and sine functions describe one revolution of the circle as their arguments vary from 0 to 2π.

$$0 \leq \omega t \leq 2\pi$$

$$0 \leq t \leq \frac{2\pi}{\omega} = \frac{1}{2} \text{ second}$$

Thus $\omega = 4\pi$, hence

$$\vec{r}(t) = 0.5\cos(4\pi t)\vec{i} + 0.5\sin(4\pi t)\vec{j}.$$

Find the velocity vector for the position of the ball at time t.

$$\begin{aligned}
\vec{v}(t) &= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{0.5\cos(4\pi(t+\Delta t))\vec{i} + 0.5\sin(4\pi(t+\Delta t))\vec{j} - [0.5\cos(4\pi t)\vec{i} + 0.5\sin(4\pi t)\vec{j}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[0.5\cos(4\pi(t+\Delta t)) - 0.5\cos(4\pi t)]\vec{i} + [0.5\sin(4\pi(t+\Delta t))\vec{j} - 0.5\sin(4\pi t)]\vec{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{0.5\cos(4\pi(t+\Delta t)) - 0.5\cos(4\pi t)}{\Delta t}\vec{i} + \frac{0.5\sin(4\pi(t+\Delta t))\vec{j} - 0.5\sin(4\pi t)}{\Delta t}\vec{j} \\
&= \lim_{\Delta t \to 0} \frac{0.5\cos(4\pi(t+\Delta t)) - 0.5\cos(4\pi t)}{\Delta t}\vec{i} + \lim_{\Delta t \to 0} \frac{0.5\sin(4\pi(t+\Delta t))\vec{j} - 0.5\sin(4\pi t)}{\Delta t}\vec{j} \\
&= \frac{d}{dt}(0.5\cos(4\pi t))\vec{i} + \frac{d}{dt}(0.5\sin(4\pi t))\vec{j} \\
&= -0.5\sin(4\pi t)\cdot 4\pi\vec{i} + 0.5\cos(4\pi t)\cdot 4\pi\vec{j}
\end{aligned}$$

Now sketch a few examples to see that this calculation makes sense.

Next find the acceleration of the ball at time t.

$$\begin{aligned}
\vec{a}(t) &= \lim_{\Delta t \to 0} \frac{\Delta \vec{v}(t)}{\Delta t} \\
&= \frac{d}{dt}(-0.5\sin(4\pi t)\cdot 4\pi)\vec{i} + \frac{d}{dt}(0.5\cos(4\pi t)\cdot 4\pi)\vec{j} \\
&= -0.5(4\pi)^2\cos(4\pi t)\vec{i} - 0.5(4\pi)^2\sin(4\pi t)\vec{j}
\end{aligned}$$

Draw a picture to show the displacement vector, velocity vector and acceleration vector all in one image. Here is a summary of what we have observed

Uniform Circular Motion: For a particle whose motion is described by

$$\vec{r}(t) = R\cos(\omega t)\vec{i} + R\sin(\omega t)\vec{j}$$

- Motion is in a circle of radius R
- One complete revolution of the circle is traversed for t-values satisfying $0 \le \omega t \le 2\pi$.
- The velocity vector, $\vec{v}(t)$, is tangent to the circle and the speed is constant $||\vec{v}|| = |\omega|R$
- The acceleration vector, $\vec{a}(t)$, points towards the center of the circle with $||\vec{a}|| = |\omega|^2 R$

- motion along a line

  Find the velocity vector for the displacement vector $\vec{r}(t) = 2\vec{i} + 5\vec{j} + (t^3 - 2t)(\vec{i} - 3\vec{j})$.

  We calculate $\vec{v}(t) = (3t^2 - 2)(\vec{i} - 3\vec{j})$. What does this tell us about the path of the displacement vector? The velocity is in the direction of $\vec{i} - 3\vec{j}$, except when $3t^2 - 2$ is negative which then gives a vector in the direction of $-(\vec{i} - 3\vec{j})$. (continue from pg. 890 HH)

  How many of the displacement vectors given below describe the position of an object moving along a linear path in the plane?

  1. $\vec{r}(t) = (4t+1)\vec{i} + (2-7t)\vec{j}$

2. $\vec{r}(t) = (4t^2+1)\vec{i} + (2-7t^2)\vec{j}$
3. $\vec{r}(t) = (4\sin(t)+1)\vec{i} + (2\sin(t)-7t)\vec{j}$

1. The first is equivalent to the parametric equation $x = 4t+1, y = 2-7t$ which is a line of slope $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-7}{4}$. The velocity vector for this displacement vector is $\vec{v}(t) = 4\vec{i} - 7\vec{j}$. Thus this object is traveling at constant velocity along a line of slope $-\frac{7}{4}$. Notice that the velocity $\vec{v}$ also lies along a line of slope $-\frac{7}{4}$.

2. The second equation is seemingly not linear because of the t^2 term, but do not rush to judgement too soon! Let's look at the velocity vector for this object. The second velocity vector is $\vec{v}(t) = 8t\vec{i} - 14t\vec{j} = t(8\vec{i} - 14\vec{j})$. Hence the velocity is always in the direction of $8\vec{i} - 14\vec{j}$.

- arc length of a curve in the plane (for a function and for a parametric curve)

Finding the length of a parametric curve. Consider the parametric equation given by $(x,y) = (f(t), g(t))$. To find the arc length between point $A = (f(a), g(a))$ and point $B = (f(b), g(b))$. To proceed, we make the standard assumptions that f and g are 'nice' functions, that is they have continuous derivatives that are not simultaneously zero.

Here are the steps. Step 1: Cut up the curve into n pieces.
Step 2: On each piece, approximate the length of the curve with the length of a straight line between the two endpoints.
Step 3: Repeat the previous two steps for larger and larger n. Does doing so make the error in the approximation smaller and smaller?! Step 4: Take the limit as $n \to \infty$.

To put these steps into action we need to look at the formula for the length of a straight line. Before that, however, we need some notation for the n pieces. The curve between A and B is described by the t-values in the interval $a \leq t \leq b$. Cutting this up into n pieces we use the notation t_i for the t-value of the endpoint of the i^{th} piece; $a = t_0 < t_1 < t_2 < ... < t_i < ... < t_n = b$. For each value of i, let the point on the curve P_i be given by $(f(t_i), g(t_i))$, then the i^{th} piece will be between P_{i-1} and P_i.

Now on to Step 2. On the i^{th} piece of the curve, we approximate the length of the curve with the length of the line segment between P_{i-1} and P_i.

$$\begin{aligned}
\text{approximate length of } i^{th} \text{ piece } &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\
&= \sqrt{[f(t_{i-1}) - f(t_i)]^2 + [g(t_{i-1}) - g(t_i)]^2} \\
&= \sqrt{[\frac{f(t_{i-1}) - f(t_i)}{\Delta t_i}\Delta t_i]^2 + [\frac{g(t_{i-1}) - g(t_i)}{\Delta t_i}\Delta t_i]^2} \\
&\approx \sqrt{[f'(t_i^*)\Delta t_i]^2 + [g'(t_i^{**})\Delta t_i]^2} \\
&= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\Delta t_i
\end{aligned}$$

Adding up all n approximations we have

$$\begin{aligned}
\text{Total Arc Length } &\approx \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\Delta t_i \\
\text{Total Arc Length } &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt
\end{aligned}$$

Example 3.2.6. Show that the circumference C of a circle of radius R is given by $C = 2\pi R$.

Example 3.2.7. Find the length of the curve $(x, y) = (\cos(t), t + \sin(t))$ on the interval $0 \le t \le \pi$.

Example 3.2.8. Find the length of the function $y = x^{3/2}$ on the interval $0 \le x \le 4$. (Derive formula for arc length of a function.)

- **Homework: Section 17.2 problems 2, 6, 10, 16, 18 (linear motion: acceleration in same direction as velocity when speed increasing, opposite direction when speed decreasing), 22, 26, good physics problems: 27, 32, 38 (have Michaela look at these and others in this section)**

3.3 Area, Moments and Center of Mass, Surface area, Pappus' Theorem

Area, Moments and Center of Mass, Surface area (for parametric curves & solids of revolving $f(x)$ on $[a, b]$, from Stewart/Thomas).

- area between curves
- area inside a loop (parametric) Perhaps Skip?
- center of mass (show video of long skinny object rotating while thrown up and over)

3.4 Polar Coordinates, from Stewart

- defn, translating to Cartesian, translating from Cartesian to Polar
- graphs of polar curves, polar regions in the plane (back to uniform circular motion parametrized by angle)
- area in polar (NEED INTEGRATION TECHNIQUES) (postpone until we have integration techniques)
- arc length in polar (optional)
- LOOK in other texts for good polar coordinate applications to physics. !! Kepler's laws, yes, but too ugly to present?! cardioid microphones.

Chapter 4

Integration Techniques and Applications

4.1 U-substitution

Outline of Topics:

- several examples emphasizing how to make good choices for u, translating limits to u-values, and how to use u-sub twice

Learning Outcomes:

- Identify integrals on which substitution can be used, and integrals where substitution does not help
- Articulate why the method of substitution is valid. Which differentiation rule does substitution come from?
- Recognize appropriate choices for u-sub, apply this choice, and evaluate the integral coerctly

4.2 Integration by Parts

Outline of Topics:

- emphasize $\displaystyle\int t^n \sin(mt)dt$, $\displaystyle\int t^n \cos(mt)dt$, $\displaystyle\int e^t \sin(mt)\cos(kt)dt$, $\displaystyle\int e^t \cos(kt)dt$, $\displaystyle\int t^n e^t dt$

Learning Outcomes:

- Identify integrals on which integration by parts can be used, and integrals where it does not help
- Articulate why the integration by parts formula is valid. Which differentiation rule does the formula come from?
- Recognize appropriate choices for u and dv, apply these choice, and evaluate the integral correctly

4.3 Trig Integrals

Outline of Topics:

- emphasize $\int \sin^2(t)dt$, $\int \cos^2(t)dt$, $\int \sin(mt)\cos(kt)dt$ (keeping Fourier series in mind)

Learning Outcomes:

- Recognize appropriate integration technique, apply this technique, and evaluate the integral correctly
- Apply trig identities appropriately to evaluate integrals

Lecture 1. (rough draft) App from Stewart (found online): Household electricity is supplied in the form of alternating current that varies from 155 V to -155 V with a frequency of 60 cycles per second (Hz). The voltage is thus given by the equation

$$E(t) = 155\sin(120\pi t)$$

where t is the time in seconds. Voltmeters read the RMS (root- mean-square) voltage, which is the square root of the average value of $[E(t)]^2$ over one cycle.

(a) Calculate the RMS voltage of household current.

(b) Many electric stoves require an RMS voltage of 220 V. Find the corresponding amplitude A needed for the voltage $E(t) = Asin(120\pi t)$.

4.4 Applications of Integration

Ask Michaela to help prioritize. Not enough time for all applications.

- area between curves (yes)
- Volume, (yes)
- Centroids (No)
- Moments of Inertia, Center of Mass (top on Michaela)
- **Average Value** (impt for Fourier series!),
- Force due to Liquid Pressure,
- Arc Length (yes)
- Surface Area (maybe)