# Cosets!

For an element $a \in G$ and a subgroup $H \leq G$, the ***left coset*** $aH$ is

$$aH = \{ah \,|\, h \in H\}$$

and the ***right coset*** $Ha$ is

$$Ha = \{ha \,|\, h \in H\}.$$

The element $a$ is a ***coset representative*** of $aH$ or $Ha$.

## Properties of cosets

Suppose that $G$ is a group, $H \leq G$ and $a, b \in G$.

1. $a \in aH$.
2. $aH = H$ if and only if $a \in H$.
3. $aH = bH$ if and only if $a \in bH$.
4. Either $aH = bH$ or $aH \cap bH = \emptyset$.
5. $aH = bH$ if and only if $a^{-1}b \in H$.
6. $|aH| = |bH|$.
7. $aH = Ha$ if and only if $H = aHa^{-1}$.
8. $aH$ is a subgroup of $G$ if and only if $aH = H$.
9. The left cosets of $H$ partition the group $G$.
10. **Lagrange's Theorem.** If $G$ is finite, and there are $r$ distinct left cosets of $H$ in $G$, then $|G| = r|H|$. In particular, $|H|$ divides $|G|$. The number $r$ is the ***index*** of $H$ in $G$ and is denoted $|G : H|$.
11. If $G$ is finite, then $|a|$ divides $|G|$.
12. If $|G|$ is prime, then $G$ is cyclic. In particular, if $|G| = p$ is prime, then $G \approx \mathbb{Z}_p$.
13. If $G$ is finite, then $a^{|G|} = e$.

## Orbits and stabilizers!

Let $G$ be a group of permutations of a set $S$.

For $i \in S$, the ***stabilizer*** of $i$ in $G$ is $\mathrm{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$.

For $i \in S$, the ***orbit*** of $i$ in $S$ is $\mathrm{orb}_G(i) = \{\phi(i) \mid \phi \in G\}$.

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**Theorem (Orbit-Stabilizer Theorem)**

*Let $G$ be a finite group of permutations of a set $S$. Then for any $i \in S$, $|G| = |\text{orb}_G(i)||\text{stab}_G(i)|$.*