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The ***factor group*** G/H is the set $\{aH \mid a \in G\}$ with operation $(aH)(bH) = abH$.

Theorem

⊛ ⊛ ⊛ *G/H is a group if and only if $H \lhd G$.* ⊛ ⊛ ⊛

Internal direct products!

G is the ***internal direct product*** of two subgroups H and K if

- H and K are both normal subgroups of G
- $G = HK = \{hk \mid h \in H, k \in K\}$
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Theorem

$H_1 \times \cdots \times H_n \approx H_1 \oplus \cdots \oplus H_n$.

Cool applications of normal subgroups, factor groups, and internal direct products

Theorem

If $G/Z(G)$ is cyclic then G is Abelian.
(And $G = Z(G)$ and $G/Z(G) = \{e\}$.)

Theorem

$G/Z(G) \approx \text{Inn}(G)$.

Theorem

If G is a finite Abelian group, p is prime, and p divides G, then G has an element of order p.

Theorem

If p is prime and $|G| = p^2$ then $G \approx \mathbb{Z}_{p^2}$ or $G \approx \mathbb{Z}_p \oplus \mathbb{Z}_p$.
(So all groups of order p^2 are Abelian.)