Homomorphisms!

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The ***kernel*** of the homomorphism ϕ is the set of elements that map to the identity of $\overline{G}$,

$$\text{Ker}\,\phi = \{g \in G \,|\, \phi(g) = e\}.$$

Properties of Homomorphisms

Let $\phi : G \to \overline{G}$ be a homomorphism, and let $H \leq G$ and $a, b \in G$.

1. $\phi(e_G) = e_{\overline{G}}$ and $\phi(a^n) = (\phi(a))^n$ for all $n \in \mathbb{Z}$.
2. $\operatorname{Ker} \phi \lhd G$.
3. $\phi(a) = \phi(b)$ if and only if $a \operatorname{Ker} \phi = b \operatorname{Ker} \phi$.
4. $\phi(a) = \overline{a}$ if and only if $\phi^{-1}(\overline{a}) = a \operatorname{Ker} \phi$.
5. If H is Abelian then $\phi(H)$ is Abelian.
6. If $H = \langle a \rangle$ then $\phi(H) = \langle \phi(a) \rangle$.
7. If $|a|$ is finite, $|\phi(a)|$ divides $|a|$.
8. $H \leq G$ if and only if $\phi(H) \leq \overline{G}$.
9. $H \lhd G$ if and only if $\phi(H) \lhd \overline{G}$.
10. If $|\operatorname{Ker} \phi| = n$ then $\phi : G \to \phi(G)$ is an n-to-1 map.
11. $\operatorname{Ker} \phi = \{e\}$ if and only if $\phi : G \to \phi(G)$ is an isomorphism.
12. **First Isomorphism Theorem.** $\alpha : G/\operatorname{Ker} \phi \to \phi(G)$ given by $\alpha(a \operatorname{Ker} \phi) = \phi(a)$ is an isomorphism.
13. $|\phi(G)|$ divides $|G|$ and $|\overline{G}|$, and $|\phi(H)|$ divides $|H|$, $|G|$, and $|\overline{G}|$.
14. If $H \lhd G$ then $\beta : G \to G/H$ given by $\beta(a) = aH$ is a homomorphism and $H = \operatorname{Ker} \beta$.