Isomorphisms

An ***isomorphism*** ϕ from a group G to a group $\overline{G}$ is a bijection $\phi : G \to \overline{G}$ that is ***operation preserving***, which means

$$\phi(ab) = \phi(a)\phi(b).$$

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Theorem

Isomorphism is an equivalence relation on groups.

Properties of Isomorphisms

Let $\phi : G \to \overline{G}$ be an isomorphism.

1. $|G| = |\overline{G}|$.
2. $\varphi(e_G) = e_{\overline{G}}$.
3. $(\varphi(a))^{-1} = \varphi(a^{-1})$ for all $a \in G$.
4. $\varphi(a^n) = (\varphi(a))^n$ for all $a \in G$ and $n \in \mathbb{Z}$.
5. a and b commute if and only if $\varphi(a)$ and $\varphi(b)$ commute.
6. G is Abelian if and only if $\overline{G}$ is Abelian.
7. $G = \langle a \rangle$ if and only if $\overline{G} = \langle \varphi(a) \rangle$.
8. $|a| = |\varphi(a)|$ for all $a \in G$.
9. For any subset $H \subseteq G$, $H \leq G$ if and only if $\varphi(H) \leq \overline{G}$.
10. $\phi(Z(G)) = Z(\overline{G})$.

Amazing things about isomorphisms

An ***automorphism*** of G is an isomorphism $\phi : G \to G$.

Theorem

The set of automorphisms of G*,* $\mathrm{Aut}(G)$*, forms a group under function composition.*

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♡ ♡ ♡ Cayley's Theorem ♡ ♡ ♡

Every group is isomorphic to a group of permutations.
In particular, every finite group is a subgroup of S_n*.*