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Comparability Invariance Results for Tolerance Orders

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Abstract. We prove comparability invariance results for three classes of ordered sets: bounded tolerance orders (equivalent to parallelogram orders), unit bitolerance orders (equivalent to point-core bitolerance orders) and unit tolerance orders (equivalent to 50% tolerance orders). Each proof uses a different technique and relies on the alternate characterization.

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1. Background

Throughout this paper we consider only finite ordered sets. We begin with background material on comparability invariance. A property or parameter of an ordered set is said to be *comparability invariant* if all orders with a given comparability graph have that property or have the same value of that parameter.

The first printed reference we know of to the phrase "comparability invariant" is in Habib's paper [11]. However Habib [Personal Communication, December 2000] tells us that he first learned the phrase in conversations with Arditti and Golumbic. The interest in comparability invariants arises from papers of Arditti [1], Gysin [10], and Trotter et al. [17] in which they show that all transitive orientations of a finite comparability graph have the same dimension, and one of Arditti and Jung [2] in which they extend the result to infinite comparability graphs. Trotter et al. attribute the question of whether all orders having the same comparability graph have the same dimension to Bogart. The paper [18] shows comparability invariance for a class of orders related to those considered in this paper. The paper

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[12] by Habib et al. has brought the idea of a comparability invariant to the attention of a broader audience and is one of the motivations for the current paper. A second motivation is a conversation between one of the authors (Trenk) and Habib in 1999.

It is natural to ask whether any "reasonable" property of an ordered set is a comparability invariant. The ordered set with elements t, b, and a_1, a_2, and a_3, with $b \prec a_i \prec t$ and no other relations has a comparability graph isomorphic with the ordered set t_1, t_2, t_3, a, and b, with $b \prec a \prec t_i$ and no other relations. However the number of maximal elements of the first ordered set is one, as is the number of minimal elements, while the number of maximal elements of the second ordered set is three, while the number of minimal elements is one. Thus neither the number of maximal elements nor the total number of maximal and minimal elements is a comparability invariant.

In this paper we show that the property of belonging to set S is a comparability invariant for $S = \{\text{bounded tolerance orders}\}$, $S = \{\text{unit bitolerance orders}\}$, and $S = \{\text{unit tolerance orders}\}$. Each of these classes has an alternate characterization: bounded tolerance orders are equivalent to parallelogram orders, unit bitolerance orders are equivalent to point-core bitolerance orders, and unit tolerance orders are equivalent to 50% tolerance orders. We review these alternate characterizations which will be used in the comparability invariance proofs in Section 2.

1.1. COMPARABILITY INVARIANCE

We next present the standard technique for proving that an ordered set property is a comparability invariant. Given a graph $G = (V, E)$, a set $A \subseteq V$ is called an *autonomous set* if every vertex in $V \backslash A$ is either adjacent to all of the vertices in A or none of the vertices in A. Autonomous sets play a key role in relating ordered sets that have the same comparability graph.

Let $P = (V, \prec_1)$ and $Q = (V, \prec_2)$ be ordered sets with the same comparability graph G. We say that Q is obtained from P by an *elementary reversal* if there is a set $A \subseteq V$ that is autonomous in G and satisfies the following:

1. A is not an independent set of G.
2. If x, y are not both in A then $x \prec_1 y$ iff $x \prec_2 y$.
3. If $x, y \in A$ then $x \prec_1 y$ iff $y \prec_2 x$.

In this process, Q is obtained from P (and vice versa) by reversing the comparabilities in A. Note that the definition of an autonomous set A in the comparability graph of an order P allows for the possibility of an element x which is above some elements of A and below others. However, the second and third conditions above imply that this is not possible in an autonomous set that participates in an elementary reversal. We record this below in a remark. In this paper we will only be concerned with autonomous sets that participate in elementary reversals, called *order autonomous sets* in [14].

Remark 1. If $Q = (V, \prec_2)$ is obtained from $P = (V, \prec_1)$ by an elementary reversal using the order autonomous set A, then the sets $Pred(A) = \{v \in V \mid x \prec_1 a \text{ for all } a \in A\}$, $Succ(A) = \{w \in V \mid a \prec_1 w \text{ for all } a \in A\}$, and $Inc(A) = \{z \in V \mid z||a \text{ for all } a \in A\}$ partition $V \backslash A$.

By the second condition in the definition of an elementary reversal, we could also use the relation $\prec_2$ of Q in defining the sets $Pred(A)$, $Succ(A)$, and $Inc(A)$.

If A is an order autonomous set and $a \in A$ is incomparable to every other element of A, then $A' = A\backslash\{a\}$ is another order autonomous set. Furthermore, Q can be obtained from P by an elementary reversal of A if and only if Q can be obtained from P by an elementary reversal of A'. When all such elements are removed from an order autonomous set, the resulting set will not be empty by the first condition of our definition. We record this as a remark.

Remark 2. If one ordered set is obtained from another by an elementary reversal, this can be achieved using an order autonomous set A in which every element of A is comparable to another element of A.

The following theorem of Gallai [7] (which appears in [16, p. 61–62]) shows that we can move between any two orders with the same comparability graph by a sequence of elementary reversals.

THEOREM 3 (Gallai). *Let $G = (V, E)$ be the comparability graph associated with distinct ordered sets $P = (V, \prec_P)$ and $Q = (V, \prec_Q)$. Then there exists a sequence of ordered sets $P_0, P_1, \dots, P_m$ so that $P_0 = P$, $P_m = Q$ and P_{i+1} is obtained from P_i by an elementary reversal for $i = 0, 1, \dots, m-1$.*

Theorem 3 allows us to show a property is comparability invariant by considering pairs of orders for which one can be obtained from the other by an elementary reversal. A corollary of Theorem 3 which will be useful to us is given below.

COROLLARY 4. *Let P and Q be finite ordered sets with the same comparability graph and let S be a class of orders. To prove that $P \in S \iff Q \in S$, it suffices to prove $P \in S \Rightarrow Q \in S$ where Q can be obtained from P by an elementary reversal.*

Proof. By Theorem 3 we need only prove $P \in S \iff Q \in S$ in the case that Q can be obtained from P by an elementary reversal. However, since the process of obtaining one order from another by an elementary reversal is symmetric, the result follows. □

1.2. CLASSES OF BOUNDED BITOLERANCE ORDERS

A *bounded bitolerance representation* $\langle \mathcal{I}, p, q \rangle$ of an order $P = (V, \prec)$ consists of a function $\mathcal{I}$ that maps each element $v \in V$ to an interval $I_v = [L(v), R(v)]$

on the real line and a pair of functions p and q that map each element v of V to points $p(v)$ and $q(v)$ in the interval I_v with $p(v) \neq L(v)$ and $q(v) \neq R(v)$. The point $p(v)$ is called the *left tolerant point* of v and $q(v)$ is called the *right tolerant point* of v. In a bounded bitolerance representation of $P = (V, \prec)$, we have $x \prec y$ if and only if $R(x) < p(y)$ and $q(x) < L(y)$. An ordered set is a *bounded bitolerance order* if it has such a representation. Given a bounded bitolerance representation, the *left tolerance* of element v is the quantity $t_l(v) = p(v) - L(v)$, and the *right tolerance* of element v is the quantity $t_r(v) = R(v) - q(v)$.

Bounded bitolerance orders are the orders of interval dimension two, as first observed in [15]. In [12] the property of having interval dimension 2 was shown to be a comparability invariant.

Another way to represent a bounded bitolerance order uses trapezoids. Let L_1 and L_2 be horizontal lines with L_1 above L_2. We consider trapezoids T_v that have one base on L_1 and the other base on L_2, and we allow degenerate trapezoids in which either or both bases is a point. For trapezoids T_v with bases on L_1 and L_2, we write $T_x \ll T_y$ when $T_x \cap T_y = \emptyset$ and every point of T_x is to the left of some point of T_y, which we shorten to "T_x is to the left of T_y". Similarly for intervals I_x, I_y we will write $I_x \ll I_y$ when $I_x \cap I_y = \emptyset$ and I_x is to the left of I_y.

A *trapezoid representation* of $P = (V, \prec)$ is a function T that assigns to each $v \in V$ a trapezoid T_v with one base on L_1 and the other base on L_2, so that $x \prec y$ iff $T_x \ll T_y$. An ordered set is called a *trapezoid order* (first defined in [5]) if it has such a representation. The next result appears in [15].

PROPOSITION 5. *An ordered set P is a bounded bitolerance order if and only if P is a trapezoid order.*

Proof. Given a bounded bitolerance representation $\langle \mathcal{I}, p, q \rangle$ of P, let the base of T_v along L_1 be the interval $[L(v), q(v)]$ and the base of T_v along L_2 be the interval $[p(v), R(v)]$. It is easy to check that this provides a trapezoid representation of P. The process is reversible. To ensure that $L(v) \leq R(v)$ and $p(v), q(v) \in I_v$ for each v in V, it may be necessary to first subtract'a constant from all points on L_1, so that the left (respectively right) endpoint of the base on L_1 is to the left of the left (respectively right) endpoint of the base on L_2. □

A *bounded tolerance order* is an order $P = (V, \prec)$ with a bounded bitolerance representation $\langle \mathcal{I}, p, q \rangle$ in which $t_l(v) = t_r(v)$ for each $v \in V$. In this case we call the quantity $t_l(v) = t_r(v)$ the *tolerance* of v and denote it by t_v. In this light, a bounded tolerance representation $\langle \mathcal{I}, t \rangle$ consists of an assignment of intervals $\mathcal{I} = \{I_v \mid v \in V\}$ and tolerances $t = \{t_v \mid v \in V\}$ for which $0 < t_v \leq |I_v|$ for all $v \in V$. It is an easy exercise to check that x and y are comparable if and only if the quantity $|I_x \cap I_v|$ is strictly less than both tolerances t_x, t_y.

When a bounded tolerance representation is converted to a trapezoid representation using the construction in the proof of Proposition 5, one can check that the resulting trapezoids are in fact parallelograms. An ordered set $P = (V, \prec)$ is a *parallelogram order* if it has a trapezoid representation in which every trapezoid is a parallelogram. This assignment of parallelograms to elements v of V provides a *parallelogram representation* of P.

The next proposition was first observed in [15] and will be useful when we prove the comparability invariance of membership in class of bounded tolerance orders. The proof is analogous to that of Proposition 5.

PROPOSITION 6. *An ordered set P is a bounded tolerance order if and only if P is a parallelogram order.*

In our proofs of comparability invariance, we need the notion of "scaling down" the (induced) representation of a subset W of V. This idea is based on a construction in [12].

Given a parallelogram representation $\{P_v \mid v \in V\}$ of an order P and a subset $W \subseteq V$, we can scale down the parallelograms in the set $\{P_w \mid w \in W\}$ as follows. Fix a sufficiently large number M and translate the parallelograms in $\{P_w \mid w \in W\}$ horizontally so that they fit between the lines $x = 0$ and $x = M$. To *scale down* the representation so that it fits between the lines $x = 0$ and $x = m$, map the point (a, b) of parallelogram P_w to the point $(\frac{am}{M}, b)$. Each parallelogram P'_w in the resulting set of parallelograms $\{P'_w \mid w \in W\}$ still has sides along L_1 and L_2 and can be translated horizontally to fit in any space of width m. Note that for all x and y in W we have $P_x \ll P_y$ iff $P'_x \ll P'_y$ However, scaling down the parallelograms in W can change their comparability with other parallelograms representing members of $V \backslash W$, and this will be taken into consideration in our proof of Theorem 10.

The second special class of bounded bitolerance orders for which we will prove a comparability invariance result is the class of unit bitolerance orders. An order $P = (V, \prec)$ is a *unit bitolerance order* if it has a bounded bitolerance representation $\langle \mathcal{I}, p, q \rangle$ in which $|I_v|$ is a constant for all $v \in V$.

An analogous scaling down of some of the intervals in a unit bitolerance representation will not result in another unit bitolerance representation. Instead we use a different representation of unit bitolerance orders which can be scaled down. A *point-core bitolerance order* is an order $P = (V, \prec)$ with a bounded bitolerance representation $\langle \mathcal{I}, p, q \rangle$ in which $p(v) = q(v)$ for each $v \in V$. In this case, we denote by $f(v)$ this *splitting point* $f(v) = p(v) = q(v) \in I_v$. Thus a point-core bitolerance representation of $P = (V, \prec)$ consists of an assignment to each $v \in V$, an interval $I_v = [L(v), R(v)]$ and a splitting point $f(v)$ in the open interval $(L(v), R(v))$. We denote the representation by $\langle \mathcal{I}, \mathcal{F} \rangle$ where $\mathcal{I} = \{I_v \mid v \in V\}$ and $\mathcal{F} = \{f(v) \mid v \in V\}$. In a point-core representation of $P = (V, \prec)$ we have $x \prec y$ iff $R(x) < f(y)$ and $f(x) < L(y)$. This leads us to make the following remark.

Remark 7. If $P = (V, \prec)$ is a point-core bitolerance order with representation $\langle \mathcal{I}, \mathcal{F} \rangle$, then the relation between two elements x and y is completely determined by the order of the points $L(x)$, $f(x)$, $R(x)$, $L(y)$, $f(y)$, $R(y)$. This allows us to convert one point-core bitolerance representation of an order P into another by perturbing endpoints and splitting points, as long as we do not change the order of these points.

Point-core bitolerance orders are called *split interval orders* in [6] and their representations are called *Fishburn representations* by [4]. Surprisingly, the classes of unit bitolerance orders and point-core bitolerance orders are equal, as shown in [15]. We sketch a proof below.

PROPOSITION 8. *A finite ordered set P is a unit bitolerance order iff P is a point-core bitolerance order.*

Proof. Given a unit bitolerance representation $\langle \mathcal{I}, p, q \rangle$ of $P = (V, \prec)$ in which $I_v = [L(v), R(v)]$, let $L'(v) = p(v)$, $R'(v) = q(v) + R(v) - L(v)$ and $f'(v) = R(v)$ for each $v \in V$. It is easy to check that the intervals I'_v and splitting points $f'(v)$ provide a point-core bitolerance representation of P. Conversely, given a point-core bitolerance representation of P with interval $I'_v = [L'(v), R'(v)]$ and splitting point $f'(v)$ assigned to v, let $L(v) = f'(v) - M$, $q(v) = R'(v) - M$, $p(v) = L'(v)$ and $R(v) = f'(v)$ where M is chosen large enough so that $L(v) < p(v)$ and $q(v) < R(v)$ for all v. Again, one can check that this provides a unit bitolerance representation of P. □

Given a point-core bitolerance representation $\langle \mathcal{I}, \mathcal{F} \rangle$ of $P = (V, \prec)$ and a subset $W \subseteq V$, we may scale down the intervals and splitting points assigned to elements of W as follows. Translate the intervals in $\{I_w \mid w \in W\}$ and the splitting points $\{f(w) \mid w \in W\}$ horizontally so that they fit in the interval $[0, M]$ for a sufficiently large number M. To *scale down* the representation of elements of W so that it fits in $[0, m]$, map the intervals and splitting points by $a \to \frac{am}{M}$ so the interval $I_w = [a, b]$ maps to $I'_w = [\frac{am}{M}, \frac{bm}{M}]$ and the splitting point $f(w)$ maps to $\frac{f(w)m}{M}$. It is easy to check that these new intervals and splitting points give another point-core bitolerance representation of $(W, \prec)$. We may translate the new representation of W so that it fits in any interval of width m.

The third special class of bounded bitolerance orders we consider are the unit tolerance orders, obtained by simultaneously imposing the "unit" and "tolerance" restrictions. The class of *unit tolerance orders* are those bounded tolerance orders $P = (V, \prec)$ with a representation $\langle \mathcal{I}, t \rangle$ in which $|I_v|$ is constant for all $v \in V$. Again, if we scale down some (but not all) of the intervals in a unit tolerance representation, we do not arrive at another unit tolerance representation. So once again we use different type of representation of the class that will allow this kind of scaling down.

The 50% *tolerance orders* are bounded tolerance orders $P = (V, \prec)$ with a representation $\langle \mathcal{I}, t \rangle$ in which $t_v = \frac{1}{2}|I_v|$ for all $v \in V$. Thus a 50% tolerance representation is a point-core bitolerance representation in which the splitting point $f(v)$ lies at the center of the interval I_v for all $v \in V$. As in a point-core bitolerance representation, $x \prec y$ iff $f(x) < L(y)$ and $R(x) < f(y)$.

The next proposition appears in [3] and can be proven using the same construction as we used in the proof of Proposition 8. In this instance the "tolerance" condition ensures that the splitting point of an interval will be located at the center of that interval.

PROPOSITION 9. *A finite ordered set P is a unit tolerance order iff it is a 50% tolerance order.*

Intervals and splitting points in a 50% tolerance order can be scaled down in the same way we scaled down intervals and splitting points in a point-core bitolerance representation. If $f(w)$ lies at the center of interval $I_w = [a, b]$, then the new splitting point $f'(w) = \frac{f(w)m}{M}$ lies at the center of the new interval $I'_w = [\frac{am}{M}, \frac{bm}{M}]$, and thus the scaled down representation is still a 50% tolerance representation.

2. Main Results

2.1. BOUNDED TOLERANCE ORDERS

Our next result and Lemma 12 are implied by the work of [13]. In that paper, the authors define *tube dimension* of an ordered set and prove that tube dimension is a comparability invariant. Their proof holds for parallelogram orders as well.

THEOREM 10. *Let P and Q be ordered sets with the same comparability graph. Then P is a bounded tolerance order iff Q is a bounded tolerance order.*

Proof. By Corollary 4, it suffices to prove the following. If P is a bounded tolerance order and Q can be obtained from P by an elementary reversal, then Q is a bounded tolerance order.

Using Proposition 6, we may fix a parallelogram representation of $P = (V, \prec)$, where each parallelogram in the representation has one side along the horizontal line L_1 and the opposite along the parallel line L_2. Let Q be the order obtained from P by an elementary reversal using the order autonomous set A. Since A is not an independent set, there exist $x, y \in A$ with $x \prec y$. Therefore in the parallelogram representation of P we have $P_x \ll P_y$.

Add the appropriate constant to each point on line L_1 so that parallelogram P_x becomes a rectangle (geometrically this is equivalent to moving the line L_1 to the left or right until P_x becomes a rectangle). This provides another parallelogram representation $\{P'_v \mid v \in V\}$ of P in which there is a rectangle R of width $\epsilon > 0$ that lies strictly between P'_x and P'_y.

As discussed in Section 1.2, we may scale down and translate the parallelogram representation of A in the horizontal direction so that it fits inside R (but each parallelogram still has sides on L_1 and L_2). Reflect these parallelograms about the vertical line bisecting R and denote by P''_a the new parallelogram assigned to $a \in A$ and let $P''_v = P'_v$ for all $v \in V \backslash A$. Let $P_1 = (V, \prec_1)$ be the ordered set with this parallelogram representation. Our goal is to show that $P_1 = Q$.

The reflection serves to reverse all the comparabilities between elements of A as desired. It remains to show that the reflection leaves all other comparabilities and incomparabilities of P intact. Since $P''_v = P'_v$ for all $v \in V \backslash A$, we need only consider pairs of elements where one element is in A and the other is not. By Remark 1, we know $V \backslash A = Pred(A) \cup Succ(A) \cup Inc(A)$.

For all $u \in Pred(A)$ we have $u \prec x$ so $P''_u = P'_u \ll P'_x \ll R$. Since the parallelograms representing elements of A are located inside R we have $P''_u \ll P''_a$ hence $u \prec_1 a$ for all $a \in A$. Similarly, for all $w \in Succ(A)$ we have $y \prec w$ so $R \ll P'_y \ll P'_w = P''_w$ and thus $a \prec_1 w$ for all $a \in A$. Finally, for all $z \in Inc(A)$, the parallelogram $P''_z = P'_z$ intersects both P'_x and P'_y and thus it intersects every line segment which lies entirely between P'_x and P'_y and has one endpoint on L_1 and the other on L_2. The left edge of P''_a is such a line segment for each $a \in A$. Thus $P''_z \cap P''_a \neq \emptyset$ for all $a \in A$, so $z \| a$ in P_1 for all $a \in A$. Therefore the new set of parallelograms gives a parallelogram representation of Q as desired. By Proposition 6, Q is a bounded tolerance order. □

2.2. UNIT BITOLERANCE AND UNIT TOLERANCE ORDERS

In this section we show that the property of being a unit bitolerance order and the property of being a unit tolerance order are comparability invariants. As mentioned before, we will use the alternate characterizations of these classes. We first introduce the material common to both proofs.

Since the orders are finite, one can show that interval endpoints and splitting points in tolerance representations can be perturbed slightly so that they are distinct. We record the relevant cases below.

Remark 11. Any point-core bitolerance order has a representation in which all interval endpoints and splitting points are distinct. Any 50% tolerance order has a representation in which interval endpoints are distinct.

LEMMA 12. *Let $P = (V, \prec)$ be a point-core bitolerance order with a representation in which element v is assigned interval $I_v = [L(v), R(v)]$ and splitting point $f(v)$. Let Q be obtained from P by an elementary reversal using the order autonomous set A. If there exist $x, y \in A$ with $x \prec y$ and $R(x) < L(y)$, then Q is a point-core bitolerance order. Moreover, if P is a 50% tolerance order then so is Q.*

Proof. Let S be the interval $[R(x), L(y)]$. Each $v \in Pred(A)$ has $v \prec x$ and thus $R(v) < f(x) < R(x)$. Similarly, each $w \in Succ(A)$ has $y \prec w$ and thus $L(y) < f(y) < L(w)$. For each $z \in Inc(A)$ the interval I_z must intersect I_x and

I_y, thus $L(z) \leq R(x)$ and $R(z) \geq L(y)$. This means that all intervals assigned to elements in $Pred(A)$ are completely to the left of S, all intervals assigned to elements in $Succ(A)$ are completely to the right of S, and all intervals assigned to elements in $Inc(A)$ completely contain S.

As discussed in Section 1.2, scale down those intervals representing elements of A so that the entire representation of A fits inside S. Next, reflect the intervals representing elements of A about the midpoint of S. The reflection serves to reverse all comparabilities in A while keeping the intervals assigned to elements in A entirely inside S. This has the desired effect of leaving all other comparabilities and incomparabilities in P intact. Thus the new set of intervals and splitting points provides a point-core bitolerance representation of Q. To justify the final sentence of the lemma, note that if the original representation of P was a 50% tolerance representation, then so is the final representation. □

LEMMA 13. *Let $P = (V, \prec)$ be a point-core bitolerance order with a representation in which element $v \in V$ is assigned the interval $I_v = [L(v), R(v)]$ and the splitting point $f(v)$. Let Q be obtained from P by an elementary reversal of the autonomous set A. Further, suppose that $R(x) \geq L(y)$ for all $x, y \in A$ with $x \prec y$. Then*

(i) *there exists an interval S with $S \subseteq I_a$ for all $a \in A$,*
(ii) *for every $v \in Pred(A)$ the interval I_v is completely to the left of S,*
(iii) *for every $w \in Succ(A)$ the interval I_w is completely to the right of S, and*
(iv) *for every $z \in Inc(A)$ we have either $S \subseteq I_z$ or $f(z) \in S$.*

Proof. By Remark 11 we may assume that the endpoints of the intervals in $\{I_v \mid v \in V\}$ are distinct. By the hypothesis that $R(x) \geq L(y)$ for all $x, y \in A$ with $x \prec y$, we have $I_u \cap I_v \neq \emptyset$ for any pair u, v of comparable elements in A. Also the intervals representing any pair of incomparable elements in A certainly have nonempty intersection. Thus $I_u \cap I_v \neq \emptyset$ for every pair $u, v \in A$. By the Helly property of intervals, there is a common intersection point for all the intervals assigned to elements of A. Since we have assumed interval endpoints are distinct, we know there exists an interval $S = [s_1, s_2]$ with $S \subseteq I_a$ for all $a \in A$. This establishes (i).

By Remark 2 we may assume that every element of A is comparable to another element of A. In particular, this means that for each $a \in A$, we have $f(a) \notin S$. If there were an element $a \in A$ for which $L(a) = s_1$ and $R(a) = s_2$ then $f(a) \in S$, contradicting our last assertion. By taking S to have maximum possible size, we may assume that there exist distinct $x, y \in A$ with $R(x) = s_2$ and $L(y) = s_1$. Furthermore since $f(x) \notin S$, we have $f(x) < s_1 = L(y)$ and since $f(y) \notin S$, we have $f(y) > s_2 = R(x)$, so $x \prec y$. Every $v \in Pred(A)$ satisfies $v \prec x$ and thus $R(v) < f(x) < s_1$. So I_v is completely to the left of S for all $V \in Pred(A)$, proving (ii). Every $w \in Succ(A)$ satisfies $y \prec w$ and thus

$s_2 < f(y) < L(w)$. So I_w is completely to the right of S for all $w \in Succ(A)$, proving (iii).

Finally, we show (iv). Assume $z \in Inc(A)$ and $S \not\subseteq I_z$. We need to show $f(z) \in S$, so for a contradiction we first assume $f(z) < s_1$. In this case, $f(z) < s_1 = L(y)$ but since $u||y$ we must have $R(z) > f(y)$. However, $f(z) < s_1 < s_2 < f(y) < R(z)$, so $S \subseteq I_z$, contradicting our original assumption. We get a similar contradiction if we assume $f(z) > s_2$. Thus $f(z) \in S$ and this establishes (iv). □

THEOREM 14. *Let P and Q be finite ordered sets with the same comparability graph. Then P is a unit bitolerance order iff Q is a unit bitolerance order.*

Proof. By Corollary 4, it suffices to prove the following: If P is a unit bitolerance order and Q can be obtained from P by an elementary reversal, then Q is a unit bitolerance order.

Using Proposition 8, fix a point-core bitolerance representation of $P = (V, \prec)$ in which $v \in V$ is assigned the interval $I_v = [L(v), R(v)]$ and splitting point $f(v)$ with $L(v) < f(v) < R(v)$. By Remark 11 we may assume that the endpoints of these intervals and the splitting points are distinct. Let $Q = (V, \prec')$ be the order which is obtained from P by an elementary reversal using the order autonomous set A. By Remark 1, the sets $Pred(A)$, $Succ(A)$, and $Inc(A)$ partition $V \backslash A$.

Case 1. There exist $x, y \in A$ with $x \prec y$ and $R(x) < L(y)$.

By Lemma 12, Q is a point-core bitolerance order, and hence a unit bitolerance order as desired.

Case 2. For all $x, y \in A$ with $x \prec y$ we have $R(x) \geq L(y)$.

In this case, Lemma 13 applies, so we know (i) there exists a real interval $S = [s_1, s_2]$ with $S \subseteq I_a$ for all $a \in A$, (ii) for every $v \in Pred(A)$ the interval I_v is completely to the left of S, (iii) for every $w \in Succ(A)$ the interval I_w is completely to the right of S, and (iv) for every $z \in Inc(A)$ we have either $S \subseteq I_z$ or $f(z) \in S$.

Now choose a point $h \in S$ which is different from all splitting points in the representation of P. Reflect each interval assigned to an element of A about h and denote the resulting interval for a by $I'_a = [L'(a), R'(a)]$ and the new splitting point by $f'(a)$. Since $S \subseteq I_a$ we have $L'(a) < h < R'(a)$ for each $a \in A$. The reflection serves to reverse all comparabilities in A. However, this reflection may affect other comparabilities between elements in A and elements in $V \backslash A$, and so we will make a further adjustment to the intervals in A. Our goal is to create new intervals $\{I''_a \mid a \in A\}$ so that each I''_a contains S and is contained in an interval slightly larger than S.

Choose $\epsilon > 0$ sufficiently small so that there are no endpoints of intervals or center points within ϵ of s_1 and s_2. Thus the intervals I_v which contain S also contain the larger interval $[s_1 - \epsilon, s_2 + \epsilon]$.

Consider the set $(L'(a) \mid a \in A\} \cup \{f'(a) \mid a \in A\} \cup \{R'(a) \mid a \in A\}$ and create a new representation by sliding these points so that (i) the ordering of these points is maintained, (ii) any point less than h ends up in the interval $[s_1 - \epsilon, s_1]$, and (iii) any point greater than h ends up in the interval $[s_2, s_2 + \epsilon]$.

By (i) and Remark 7, this will not disturb the comparabilities among elements in A. The new interval assigned to $a \in A$, denoted by I''_a, will contain S for each $a \in A$, and is contained in the slightly larger interval $[s_1 - \epsilon, s_2 + \epsilon]$.

Now for each $v \in Pred(A)$ and each $a \in A$ we have $I_v \ll I''_a$, thus $v \prec a$ in the new representation. For each $w \in Succ(A)$ we have $I''_a \ll I_w$ for each $a \in A$, thus $a \prec w$ in the new representation. For each $z \in Inc(A)$ we have either $I''_a \subseteq [s_1 - \epsilon, s_2 + \epsilon] \subseteq I_z$ or $f(z) \in S \subseteq I''_a$ for each $a \in A$. In either case, $z||a$ in the new representation. So the original intervals and splitting points for elements in $V \backslash A$ together with the new intervals and splitting points for elements of A gives a point-core bitolerance representation of Q. By Proposition 8, Q is a unit bitolerance order. □

THEOREM 15. *Let P and Q be finite ordered sets with the same comparability graph. Then P is a unit tolerance order iff Q is a unit tolerance order.*

Proof. By Corollary 4, it suffices to prove the following: if P is a unit tolerance order and Q can be obtained from P by an elementary reversal, then Q is a unit tolerance order.

We proceed by induction. The theorem is easy to check for orders with three or fewer elements. Assume the result is true for orders with fewer than n elements, and let $P = (V, \prec)$ be a unit tolerance order with $|X| = n$. Let Q be the order which is obtained from P by an elementary reversal using the order autonomous set A.

Fix a 50% tolerance representation of P in which $v \in V$ is assigned interval $I_v = [L(v), R(v)]$ with splitting point $f(v) = \frac{1}{2}(L(v) + R(v))$ and tolerance $t_v = f(v) - L(v) = R(v) - f(v) = \frac{1}{2}|I_v|$. By Remark 11, we may assume that the endpoints of these intervals are distinct.

By definition of a 50% tolerance representation, $x \prec y$ iff $f(x) < L(y)$ and $R(x) < f(y)$. By Remark 1, the sets $Pred(A)$, $Succ(A)$, and $Inc(A)$ partition $V \backslash A$.

Case 1. There exist $x, y \in A$ with $x \prec y$ and $R(x) < L(y)$.

By Lemma 12, Q is a 50% tolerance order, and hence a unit tolerance order as desired.

Case 2. For all $x, y \in A$ with $x \prec y$ we have $R(x) \geq L(y)$.

In this case, Lemma 13 applies, so we know (i) there exists a real interval $S = [s_1, s_2]$ with $S \subseteq I_a$ for all $a \in A$, (ii) for every $v \in Pred(A)$ the interval I_v is completely to the left of S, (iii) for every $w \in Succ(A)$ the interval I_w is completely to the right of S, and (iv) for every $z \in Inc(A)$ we have either $S \subseteq I_z$ or $f(z) \in S$.

Partition $Inc(A)$ as $I_C(A) \cup I_N(A)$, where $I_C(A) = \{z \in Inc(A) \mid S \subseteq I_z\}$ is the set of elements incomparable to A whose intervals "cover" S and $I_N(A) = \{u \in Inc(A) \mid S \not\subseteq I_u\}$ is the set of elements incomparable to A whose intervals do *not* "cover" S. By condition (iv) we know $f(u) \in S$ for all $u \in I_N(A)$. If $I_N(A) = \emptyset$ then the argument in the second paragraph of the proof of Lemma 12 applies and we conclude that Q is a unit tolerance order.

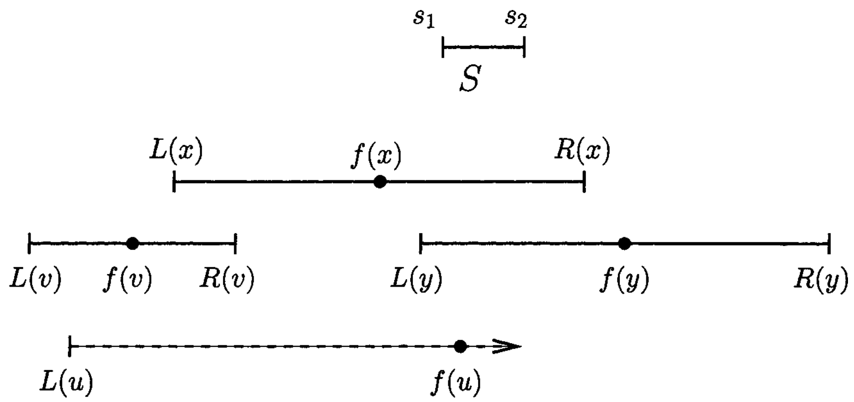

Figure 1. A figure to accompany the proof of part (b) of the claim.

Otherwise $I_N(A) \neq \emptyset$. As noted above, $f(u) \in S$, that is, $s_1 \leq f(u) \leq s_2$ for all $u \in I_N(A)$. Here our proof diverges from that of Theorem 14 since we can not slide the splitting points $f(a)$ without disturbing the property that they lie in the centers of their intervals.

CLAIM. *The set $A \cup I_N(A)$ is an order autonomous set of P.*

Proof. To prove the claim it suffices to show that for any $u \in I_N(A)$ and any $v \in V\backslash(A \cup I_N(A))$, the relation between u and v is the same as the relation between a and v for any $a \in A$. Thus for $z \in I_c(A)$, $v \in Pred(A)$, and $w \in Succ(A)$ we will show

(a) $u||z$,
(b) $v \prec u$, and
(c) $u \prec w$.

Fix elements $u \in I_N(A)$, $z \in I_C(A)$, $v \in Pred(A)$, $w \in Succ(A)$, and in addition fix elements $x, y \in A$ with $x \prec y$. To prove (a) we note that $f(u) \in S \subseteq I_z$, so $u||z$.

We next prove (b). By the definition of $S = [s_1, s_2]$ we know $L(x), L(y) \leq s_1$ and $s_2 \leq R(x), R(y)$ (see Figure 1). Since $x \prec y$ we have $f(z) < L(y) \leq s_1$ and $s_2 \leq R(x) < f(y)$. Also, since $v \in Pred(A)$ and $x \in A$ we have $v \prec x$ so

$$R(v) < f(x) < L(y) \leq s_1 < f(u). \tag{1}$$

We wish to show $f(v) < L(u)$, which together with $R(v) < f(u)$ from (1) would imply $v \prec u$ and prove (b). Suppose for a contradiction that $L(u) \leq f(v)$ (as shown by the dashed line of I_u in Figure 1). Since $v \prec x$ we have $f(v) < L(x)$ so $L(u) < L(x)$. By (1) we have $f(x) < f(u)$. Because we have a 50% tolerance representation, the splitting points $f(x)$ and $f(u)$ lie at the centers of their respective intervals, so $\frac{1}{2}(L(x) + R(x)) = f(x) < f(u) = \frac{1}{2}(L(u) + R(u)) < \frac{1}{2}(L(x) + R(u))$, and thus $R(u) > R(x)$. But then $L(u) < L(y) \leq s_1$ and

$s_2 \leq R(x) < R(u)$, which means $S \subseteq I_u$, contradicting the fact that $u \in I_N(A)$. This completes the proof of (b). A similar argument shows (c) and finishes the proof of the claim.

Case 2a. $V \neq A \cup I_N(A)$. In this case, the order P_1, induced in P by the elements in $A \cup I_N(A)$ is a 50% tolerance order with fewer than n elements. Furthermore, A is an order autonomous set in P_1. By the induction hypothesis we may fix a 50% tolerance representation of the order resulting from P_1 by reversing all comparabilities in A. As discussed in Section 1.2, scale down and translate this representation so that it fits entirely in S and place it there. This representation captures the comparabilities and incomparabilities between elements of $A \cup I_N(A)$ in Q. We leave the intervals representing elements of $V \backslash (A \cup I_N(A))$ intact, so our representation gives the correct order relations between elements of $V \backslash (A \cup I_N(A))$ in Q. By (ii), (iii) and the definition of $I_c(A)$, our representation also realizes the comparabilities and incomparabilities between elements of $A \cup I_N(A)$ and elements of $V \backslash (A \cup I_N(A))$ in Q. Therefore, Q is a 50% tolerance order and by Proposition 9, Q is a unit tolerance order.

Case 2b. $V = A \cup I_N(A)$. In this case, reflect each interval in A about the midpoint of S in order to reverse the comparabilities in A. Let I'_a be the new interval assigned to $a \in A$. Recall that $S \subseteq I_a$ for all $a \in A$ by condition (i) of case 2. Since I'_a results from reflecting I_a about the midpoint of S, we also have $S \subseteq I'_a$ for all $a \in A$. By condition (iv) of case 2, we have $f(u) \in S$ for all $u \in I_N(A)$, thus $u||a$ for all $a \in A$ as desired. This new representation is a 50% tolerance representation of Q when $V = A \cup I_N(A)$, so by Proposition 9, Q is a unit tolerance order. □

3. Conclusion

Classes of tolerance orders are often studied from the perspective of graph theory and indeed the original definition in [8] is in terms of graphs. In our notation, a graph $G = (V, E)$ is a bounded bitolerance graph if there is a bounded bitolerance representation of an order $P = (V, \prec)$, where G is the incomparability graph of P. Thus if G is a bounded bitolerance graph, then by definition *there exists* an order P, where G is the incomparability graph of P and P is a bounded bitolerance order. Once we know that membership in the class of bounded bitolerance orders is a comparability invariant, we can make the stronger statement that *every* order P whose incomparability graph is G is a bounded bitolerance order. Similar reasoning applies to the three classes of orders about which we have proven comparability invariance results in this paper.

For a comprehensive treatment of tolerance graphs and orders, the reader is refered to [9].

References

1. Arditti, J. C. (1976) Graphes de comparabilité et dimension des ordres, Note de Recherches CRM 607, Centre de Recherche Mathématiques, Université Montréal.

2. Arditti, J. C. and Jung, H. A. (1980) The dimension of finite and infinite comparability graphs, *J. London Math. Soc.* **21**, 31–38.
3. Bogart, K., Fishburn, P., Isaak, G. and Langley, L. (1995) Proper and unit tolerance graphs, *Discrete Appl. Math.* **60**, 37–51.
4. Bogart, K. P. and Isaak, G. (1998) Proper and unit bitolerance orders and graphs, *Discrete Math.* **181**, 37–51.
5. Dagan, I., Golumbic, M. C. and Pinter, R. Y. (1988) Trapezoid graphs and their coloring, *Discrete Appl. Math.* **21**, 35–46.
6. Fishburn, P. C. and Trotter, W. T. (1999) Split semiorders, *Discrete Math.* **195**, 111–126.
7. Gallai, T. (1967) Transitiv orientierbare graphen, *Acta Math. Hungar.* **18**, 25–66.
8. Golumbic, M. C. and Monma, C. L. (1982) A generalization of interval graphs with tolerances, *Congressus Numerantium* **35**, 321–331.
9. Golumbic, M. C. and Trenk, A. N., *Tolerance Graphs*, monograph in preparation.
10. Gysin, R. (1977) Dimension transitive orientierbar graphen, *Acta Math. Acad. Sci. Hungar.* **29**, 313–316.
11. Habib, M. (1984) Comparability invariants, *Ann. Discrete Math.* **23**, 371–386.
12. Habib, M., Kelly, D. and Möhring, R. (1991) Interval dimension is a comparability invariant, *Discrete Appl. Math.* **88**, 211–229.
13. Habib, M., Kelly, D. and Möhring, R. (1992) Comparability invariance of geometric notions of order dimension, manuscript.
14. Kelly, D. (1986) Invariants of finite comparability graphs, *Order* **3**, 155–158.
15. Langley, L. (June 1993) *Interval Tolerance Orders and Dimension*, PhD Thesis, Dartmouth College.
16. Trotter, W. T. (1992) *Combinatorics and Partially Ordered Sets*, Johns Hopkins University Press, Baltimore, MD.
17. Trotter, W. T., Moore, J. I. and Sumner, D. (1976) The dimension of a comparability graph, *Discrete Math.* **16**, 361–381.
18. Niederle, J. (2000) Being a proper trapezoid ordered set is a comparability invariant, *Order* **17**, 301–308.