Critical Pebbling Numbers of Graphs

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Abstract

We define three new pebbling parameters of a connected graph G, the r-, g-, and u-*critical pebbling numbers*. Together with the pebbling number, the optimal pebbling number, the number of vertices n and the diameter d of the graph, this yields 7 graph parameters. We determine the relationships between these parameters. We investigate properties of the r-critical pebbling number, and distinguish between greedy graphs, thrifty graphs, and graphs for which the r-critical pebbling number is 2^d.

Key words: pebbling, optimal pebbling number, critical pebbling number

1 Pebbling Numbers

Let G be a connected graph. A *pebbling distribution* (or simply *distribution*) D on G is a function which assigns to each vertex of G a natural number of pebbles. If D is a distribution on a graph G and a is a vertex of G, we denote by $D(a)$ the number of pebbles on a in the distribution D. The *size* of the distribution D is the number of pebbles in D, $|D| = \sum_{a \in V(G)} D(a)$.

A *pebbling step* $[a, b]$ is an operation which takes the distribution D, removes two pebbles from the vertex a, and adds one pebble at the adjacent vertex b. A distribution D is *r-solvable* if there exists a sequence of pebbling steps starting with D and ending with at least one pebble on the vertex r, and *solvable* if D is r-solvable for all r. A distribution D is *unsolvable* if there is some vertex r for which D is not r-solvable. The *pebbling number* $p(G)$ of G is the minimum number such that any distribution on G with $p(G)$ pebbles is solvable [Hur99]. For example, since the first distribution on the graph C_7 in Figure 1 is unsolvable, $p(C_7) > 10$. In fact, $p(C_7) = 11$ [Hur99].

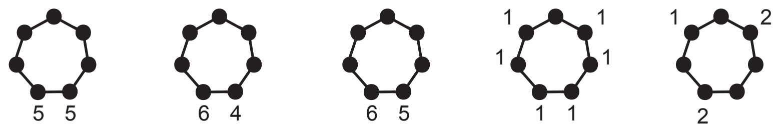

Fig. 1. Five pebbling distributions on the graph C_7.

A *rooted distribution* is a distribution which also identifies a vertex r of G as the *root vertex* of G. As before, we say that the rooted distribution D is *solved* if it has at least one pebble on r, and *solvable* if there exists a sequence of pebbling steps starting with D and ending with a solved distribution. If such a sequence exists, we call it a *solution* of D. Note that for a rooted distribution, the terms solvable and r-solvable are interchangeable. In general, any statement about a distribution D on a graph G can applied to a corresponding rooted distribution E as well, obtained from D by choosing a root of G. For emphasis, we say that an un-rooted distribution is a *global* distribution.

A rooted distribution D is *minimally r-solvable* if D is r-solvable but the removal of any pebble makes D not r-solvable. A rooted distribution D is *maximally r-unsolvable* if D is not r-solvable but the addition of any pebble makes D r-solvable. A global distribution D is *minimally solvable* if D is solvable but the removal of any pebble makes D unsolvable. A global distribution

D is *maximally unsolvable* if D is unsolvable but the addition of any pebble makes D solvable.

For example, the first distribution in Figure 2 is minimally solvable as a global distribution, since the deletion of any pebble makes it unsolvable to some root. However, it is not minimally r-solvable for any choice of a root r, since once a root is selected four pebbles can be deleted from the distribution while keeping it solvable.

Fig. 2. Two pebbling distributions on the graph $K_{2,3}$.

The eight combinations of largest or smallest, solvable or unsolvable, and rooted or global distributions yield the following five pebbling-related parameters on a graph. Two pairs of these combinations yield the same parameter, and one combination turns out to be trivial, as noted below. The remaining parameters are all different, as shown in Table 1.

- The *pebbling number* $p(G)$ is one greater than the largest size of a maximally unsolvable global distribution on G. Equivalently, $p(G)$ is one greater than the largest size of a maximally r-unsolvable rooted distribution on G for any r.

  For example, the first distribution in Figure 1 is maximally unsolvable, since the addition of any one pebble results in a solvable distribution. Since there are no maximally unsolvable distributions on C_7 with 11 pebbles, $p(C_7) = 11$.
- The *gu-critical pebbling number* $c_{gu}(G)$ is one greater than the smallest size of a maximally unsolvable global distribution on G. The *ru-critical pebbling number* $c_{ru}(G)$ is one greater than the smallest size of a maximally r-unsolvable rooted distribution on G for any r. These two parameters are equal, as proven in Lemma 1. Consequently, we define $c_u(G) = c_{ru}(G) = c_{gu}(G)$.

  For example, the fourth distribution in Figure 1 is maximally unsolvable, and there are no maximally unsolvable distributions on C_7 with fewer than 6 pebbles. So $c_u(C_7) = 7$.
- The *g-critical pebbling number* $c_g(G)$ is the largest size of a minimally solvable global distribution on G.

For example, the second distribution in Figure 1 is minimally solvable, since the deletion of any pebble makes it unsolvable. Since there are no minimally solvable distributions on C_7 with greater than 10 pebbles, $c_g(C_7) = 10$. In particular, the third distribution in Figure 1 is not minimally solvable, since removing a pebble from the vertex with 5 pebbles results in a solvable distribution.

- The *r-critical pebbling number* $c_r(G)$ is the largest size of a minimally r-solvable rooted distribution on G for any r. If a minimally r-solvable rooted distribution on G has $c_r(G)$ pebbles, then we call it an *r-ceiling* distribution.

  For example, the second distribution in Figure 2 is minimally r-solvable. Since there is no minimally r-solvable rooted distribution on $K_{2,3}$ with greater than 4 pebbles, $c_r(K_{2,3}) = 4$. In particular, as discussed above, the first distribution in Figure 2 is not minimally r-solvable for any r.
- The *optimal pebbling number* $o(G)$ is the smallest size of a minimally solvable global distribution on G [PSV95].

  For example, the fifth distribution in Figure 1 is minimally solvable, and there are no minimally solvable distributions on C_7 with fewer than 5 pebbles. So $o(C_7) = 5$.
- The smallest size of a minimally r-solvable distribution on G is 1 for any connected graph G, so we do not consider it.

The following is another helpful way of thinking of these pebbling parameters. Consider the set of all global distributions on a given graph G. Given the distributions D and E, we say that $D \leq E$ if $D(a) \leq E(a)$ for all vertices a in G. With this ordering, the set of all distributions on G becomes a lattice. Also note that if $D \leq E$ then $|D| \leq |E|$.

Now divide the distributions in this lattice into the subset S of solvable distributions and the subset U of unsolvable distributions, and also consider the set M of maximal unsolvable distributions and the set m of minimal solvable distributions. The pebbling number, g-critical pebbling number, u-critical pebbling number, and optimal pebbling number can be viewed as maxima and minima of these subsets. Specifically:

- The pebbling number is one greater than the largest size of any distribution in M.
- The u-critical pebbling number is one greater than the smallest size of any distribution in M.

- The g-critical pebbling number is the largest size of any distribution in m.
- The optimal pebbling number is the smallest size of any distribution in m.

Figure 3 shows a schematic drawing of these maxima and minima.

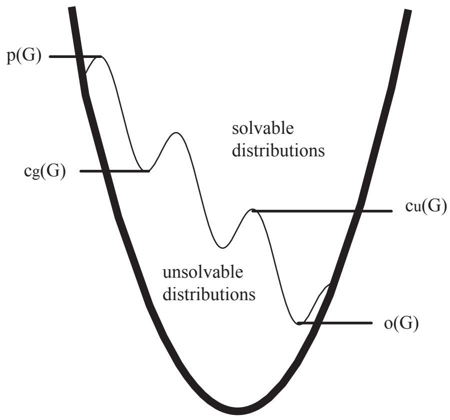

Fig. 3. The lattice of pebbling distributions of a graph.

The r-, g-, and u-critical pebbling numbers have not been previously studied. We now investigate the relationships between the five distinct pebbling numbers defined above.

Lemma 1 *For any graph G, $c_{ru}(G) = c_{gu}(G)$.*

Proof. By definition, every maximally unsolvable distribution on G is maximally r-unsolvable for some r.

Let r be a vertex of G. It suffices to show that every maximally r-unsolvable distribution is maximally unsolvable. Let D be a maximally r-unsolvable distribution on G, and assume by way of contradiction that D is not maximally unsolvable. Since D is r-unsolvable, D is unsolvable. Since D is not maximally unsolvable, there exists a vertex s of G such that D is s-unsolvable, and a pebble can be added to D so that D remains s-unsolvable. Hence, $s \neq r$.

Consider the distribution E obtained from D by adding a pebble on s. Since D is s-unsolvable, no second pebble from E can be moved to s. So any solution of E to r does not use the pebble on s. But this means that E is not r-solvable, which contradicts the fact that D is maximally r-unsolvable. □

Lemma 2 *Suppose that G is a graph with n vertices and diameter d. Then*

(1) $o(G) \leq 2^d$.

(2) $2^d \leq c_r(G)$.
(3) $c_r(G) \leq c_g(G)$.
(4) $o(G) \leq c_u(G)$.
(5) $c_u(G) \leq n$.
(6) $n \leq c_g(G)$.
(7) $c_g(G) \leq p(G)$.

Proof.

(1) The proof appears in [PSV95].

(2) Let a and b be two vertices which are distance d apart. Let D be the rooted distribution with $a = r$ and $D(b) = 2^d$. This distribution is minimally r-solvable, and consequently $2^d \leq c_r(G)$.

(3) **Case 1: There exists an r-ceiling distribution D on G which is not solvable.** Suppose D is not solvable to the vertex s. Consider the distribution D_1 obtained from D by adding a pebble at s. Since D is not solvable to s, the new pebble cannot be used in a solution of D_1 to r. Hence if we remove any pebble in D_1, either D_1 is no longer solvable to r, or D_1 is no longer solvable to s. Either D_1 is solvable, or D_1 is not solvable to some vertex t. In this second case we form the distribution D_2 by adding a pebble at t to D_1. Continuing in this way, we eventually arrive at a minimally solvable distribution E which contains all the pebbles in D and some additional pebbles. Since $c_r(G) = |D|$ and $c_g(G) \geq |E|$, $c_r(G) \leq c_g(G)$.

Case 2: All r-ceiling distributions on G are solvable. Every r-ceiling distribution is minimally r-solvable, and hence minimally solvable. Since $c_r(G)$ is the maximum size of an r-ceiling distribution and $c_g(G)$ is the maximum size of a minimally solvable distribution, $c_r(G) \leq c_g(G)$.

(4) Because $c_u(G)$ is one larger than the size of a maximally unsolvable distribution, there exists a solvable distribution with $c_u(G)$ pebbles. Since $o(G)$ is the size of the smallest solvable distribution on G, $o(G) \leq c_u(G)$.

(5) Any distribution on the graph G with one pebble on all but one vertex is maximally unsolvable. Since $c_u(G)$ is one greater than the smallest such distribution, $c_u(G) \leq n$.

(6) The distribution with one pebble on every vertex of G has n pebbles and is minimally solvable. Since $c_g(G)$ is the size of the largest such distribution, $n \leq c_g(G)$.

(7) By the definition of $p(G)$, every distribution with $p(G)$ or more pebbles is

solvable. By the definition of $c_g(G)$, there exist distributions with $c_g(G)-1$ pebbles which are not solvable. □

	K_5	$K_{2,3}$	C_7
$p(G)$	5	5	11
$c_g(G)$	5	5	10
$c_r(G)$	2	4	10
2^d	2	4	8
n	5	5	7
$c_u(G)$	5	4	7
$o(G)$	2	3	5

Table 1. Some pebbling numbers.

Table 1 shows the values of the five pebbling parameters for three graphs. The table illustrates that the inequalities in Lemma 2 can be either strict or not, and that there is no definite relationship between the remaining pairs of pebbling parameters. For instance, $c_r(K_5) < c_u(K_5)$, but $c_r(C_7) > c_u(C_7)$. The proof that $p(C_7) = 11$ appears in [Hur99], and we prove $c_r(C_7) = 10$ in Lemma 17 below. Since the remainder of the paper focuses on $c_r(G)$, we leave the rest of the values for the reader to verify.

We summarize the relationships between these seven values in the lattice in Figure 4. In this figure, an upward-slanting edge indicates that the lower value is less than or equal to the upper value for all graphs, and a missing edge indicates that each value may be greater than the other on some graphs. Finally, note that for the graph with a single vertex, all seven values are equal.

2 The r-Critical Pebbling Number

For the remainder of the paper, we focus on the r-critical pebbling number. We say that a minimally r-solvable rooted distribution is an *r-critical distri-*

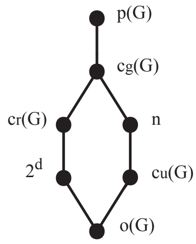

Fig. 4. The relationships between the five pebbling parameters.

bution, so $c_r(G)$ is the maximum size of an r-critical distribution on G. Recall that the r-critical distributions on G with $c_r(G)$ pebbles are called *r-ceiling distributions.*

We say that the rooted distribution D is *r-excessive* if D is r-solvable and not r-critical, and *r-insufficient* if D is not r-solvable. Then the sets of r-insufficient, r-critical, and r-excessive distributions on G form a partition of all rooted distributions on G. Note that for an r-insufficient distribution I, an r-critical distribution C, and an r-excessive distribution E, we may have $|E| < |C| < |I|$. Examples of three such distributions are shown in Figure 5.

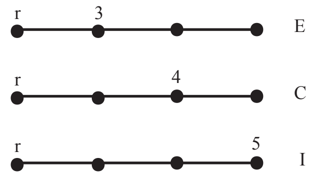

Fig. 5. Three rooted distributions on the graph P_4.

We say that a solution of the rooted distribution D is *r-critical* if it leaves one pebble on r and no pebble on any other vertex.

Lemma 3 *A rooted distribution D is r-critical if and only if D is r-solvable and all solutions of D are r-critical.*

Proof. Suppose D is r-critical. Then by definition D is r-solvable. Suppose there exists a solution S of D which is not r-critical. Then S leaves a pebble on the non-root vertex a. If this pebble is unmoved from D, we may delete it from D and obtain an r-solvable rooted distribution, which contradicts the fact that D is r-critical. So S must include the pebbling step $[b, a]$ for some other vertex b. Again, if a pebble on b is unmoved from D until this pebbling step, then D would not be r-critical. We continue in this way. Since S is a finite sequence of pebbling steps, eventually we will find a pebble in D which may be deleted to obtain an r-solvable rooted distribution. Hence, all solutions of D are r-critical.

Conversely, suppose D is r-solvable and all solutions of D are r-critical. Let E be a rooted distribution obtained by removing a pebble from D, and suppose that E is r-solvable. Then there exists a solution of D which leaves this pebble unmoved. So this solution is not r-critical, which is a contradiction. Therefore, D must be r-critical. □

Corollary 4 *If D is r-critical and a is a vertex of G with degree 1 distinct from r, then $D(a)$ is even.*

Proof. Suppose $D(a)$ is odd, and S is a solution of D. If $[b, a]$ is a pebbling step in S, then $[a, b]$ must also be a pebbling step in S. Removing both pebbling steps from S results in a non-r-critical solution of D, so D is not r-critical by Lemma 3. Alternatively, if there are no pebbling steps in S of the form $[b, a]$, then S leaves at least one pebble on a. So S is not r-critical, and again by Lemma 3, D is not r-critical. □

Lemma 5 *Suppose that G is a graph and a is a vertex of G which is adjacent to every other vertex of G. Then the r-critical distributions on G must have one of the following forms:*

(1) One pebble on r, and no pebble on any other vertex.
(2) Two pebbles on a, and no pebble on any other vertex.
(3) Four pebbles on some vertex $b \neq a$, and no pebble on any other vertex.
(4) Two pebbles on two vertices b and c, both different from a, and no pebble on any other vertex.
(5) Two pebbles on some vertex $b \neq a$, and less than two pebbles on all other vertices.

Proof. Suppose D is an r-critical distribution on G, and b and c are vertices in G other than a. If D has either one pebble on r, or two pebbles on a, or four pebbles on b, or two pebbles on both b and c, then D must have no pebbles on any other vertex. Alternatively, suppose that none of these conditions are met. In this case, D has no pebbles on r, less than two pebbles on a, less than four pebbles on any other vertex, and more than one pebble on at most one vertex.

If D has more than one pebble on no vertex, then D is r-insufficient. So without loss of generality, $2 \leq D(b) < 4$ and $D(v) < 2$ for all other vertices v in G. If $D(b) = 3$ then any solution of D will leave at least one pebble on b, and D will not be r-critical. So D must have form 5. We have shown that all r-critical distributions on G must have one of the five forms listed. $\square$

Theorem 6 *The star $K_{1,n}$ has pebbling number $n+2$ and r-critical pebbling number 4 for $n \geq 4$.*

Proof. The fact that $p(K_{1,n}) = n+2$ is a corollary of Theorem 4 of [Moe92], which gives a formula for the pebbling number of any tree.

By Lemma 2, $c_r(K_{1,n}) \geq 4$. By Lemma 5, the only r-critical distributions on $K_{1,n}$ with more than four pebbles must have two pebbles on one vertex, and one pebble on at least three other vertices. But by Lemma 4, there is only one vertex in $K_{1,n}$ which can have one pebble in an r-critical distribution. Hence $c_r(K_{1,n}) = 4$. $\square$

Note that Lemma 2 gives us $c_r(G) \leq p(G)$ for any graph G. But in fact, Theorem 6 gives an example of a family of graphs for which the difference $p(G) - c_r(G)$ is arbitrarily large.

Recall the *fan* F_k, the path P_k on k vertices with an additional vertex x adjacent to every vertex in P_k. F_8 is shown in Figure 6.

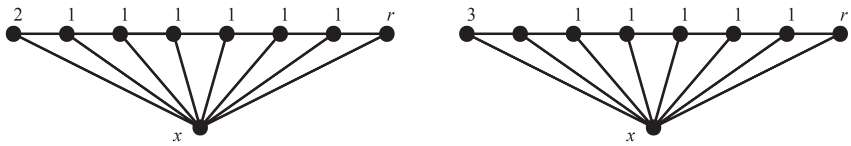

Fig. 6. An r-ceiling distribution and an r-insufficient distribution on the fan F_8.

Theorem 7 *The fan F_k has pebbling number $k+1$ for $k \geq 4$.*

Proof. Consider a rooted distribution D on F_k with $k+1$ pebbles. If D has at least 4 pebbles on any vertex, or at least 2 pebbles on each of any two vertices, or at least 2 pebbles on x, or at least 2 pebbles on any vertex and one pebble on x, then D is r-solvable. The only remaining case is that D has 3 pebbles on one vertex and one pebble on $k-2$ other vertices, which are neither x nor r. So D has pebbles on every vertex of the path $F_k - x$ other than r, with 3 pebbles on one of them, so D is again r-solvable. This implies that $p(F_k) \leq k+1$.

The second rooted distribution shown in Figure 6 has k pebbles and is r-insufficient, and consequently $p(F_k) = k + 1$. $\square$

Theorem 8 *The fan F_k has r-critical pebbling number k for $k \geq 4$.*

Proof. We first prove that $c_r(F_k) \leq k$. Let D be an r-critical distribution on F_k with more than k pebbles. Since $k \geq 4$, by Lemma 5, D must have two pebbles on one vertex and one pebble on at least $k-1$ other vertices. Therefore, either r has a pebble on it, or $r = x$, or x has a pebble on it. In each of these cases, one can easily check that D is r-excessive. Thus, there are no r-critical distributions on F_k with more than k pebbles.

The first rooted distribution shown in Figure 6 is r-critical as long as $k \geq 4$, and has k pebbles. It follows that $c_r(F_k) = k$ for $k \geq 4$. $\square$

Corollary 9 *There exist graphs with r-critical pebbling number k for all positive integers $k \neq 3$.*

Proof. For $k = 1, 2$, the path P_k has r-critical pebbling number k. For $k \geq 4$, the fan F_k has r-critical pebbling number k by Theorem 8.

Now suppose that D is a rooted distribution on the graph G with 3 pebbles. We show that D is not an r-ceiling distribution. If there is no vertex a with $D(a) \geq 2$, then there are no legal pebbling steps from D, and D is r-insufficient. If there is such a vertex, and a is adjacent to every other vertex of G, then two pebbles from a can be used to pebble to r, and D is r-excessive. If a is not adjacent to some other vertex of G, then $d(G) \geq 2$, so $c_r(G) \geq 4$. Hence, there is no r-ceiling distribution with three pebbles. $\square$

Also note that F_k is an example of a graph with diameter 2 and arbitrarily large r-critical pebbling number. Thus, $c_r(G)$ is not bounded above by any function of $d(G)$. So, although lower bounds for $c_r(G)$ do not involve n, it seems that upper bounds for $c_r(G)$ must.

3 Greed, Thrift, and Weight

We denote the distance between the vertices a and b by $d(a, b)$. The pebbling step $[a, b]$ is *greedy* if $d(a, r) > d(b, r)$; in other words, the step moves towards the root. The rooted distribution D is *greedy* if there is a solution of D which uses only greedy pebbling steps. The graph G is *greedy* if every distribution with at least $p(G)$ pebbles is greedy [Hur99]. The graph G is *thrifty* if every r-critical distribution with at least $c_r(G)$ pebbles (i.e. every r-ceiling distribution) is greedy.

In general, the pebbling number of a graph is algorithmically hard to compute. However, Hurlbert notes that for greedy graphs, the computation becomes much easier [Hur99]. For thrifty graphs, the r-critical pebbling number is even easier to compute. As we shall see below, the r-critical pebbling number of a thrifty graph is determined by its diameter.

The *weight* of the rooted distribution D is the value

$$w(D) = \sum_{v \in V(G)} \frac{D(v)}{2^{d(v,r)}}.$$

The *weight* $w(G)$ of G is the largest weight of any r-ceiling distribution on G. Note that there may be r-critical distributions with fewer pebbles and larger weight, for example, given any rooted graph G, the distribution with two pebbles on one vertex adjacent to r has weight 1. We are interested only in the weight of r-critical distributions with exactly $c_r(G)$ pebbles.

Lemma 10 *If the rooted distribution E is obtained from the rooted distribution D by a greedy pebbling step, then $w(E) = w(D)$. If the rooted distribution E is obtained from the rooted distribution D by a non-greedy pebbling step, then $w(E) < w(D)$.*

Proof. Suppose E is obtained from D by the pebbling step $[a, b]$. If $[a, b]$ is greedy and $d(a, r) = s$, then $d(b, r) = s - 1$. E has two fewer pebbles on a and one additional pebble on b. So $w(E) = w(D) - \frac{2}{2^s} + \frac{1}{2^{s-1}} = w(D)$.

If $[a, b]$ is not greedy and $d(a, r) = s$, then $d(b, r) = t \geq s$. So $w(E) = w(D) - \frac{2}{2^s} + \frac{1}{2^t} < w(D)$. □

Lemma 11 *If D is r-critical and greedy then $w(D) = 1$.*

Proof. We proceed by induction. The only r-critical distribution on G with size 1 is the rooted distribution with one pebble on r and no pebble on any other vertex. This rooted distribution is r-critical, greedy, and has weight 1 on every graph G.

Suppose that D is an r-critical, greedy distribution on G with k pebbles. Since D is greedy, there exists a greedy pebbling step $[a, b]$ from D to a new rooted distribution E with $k-1$ pebbles which is the first pebbling step in a greedy solution of D.

If E is not r-critical, then by Lemma 3 there exists a non-r-critical solution of E. Appending the pebbling step $[a, b]$ to this solution yields a non-r-critical solution of D. Therefore, E is r-critical. Because $[a, b]$ is the first step in a greedy solution of D, the remainder of this solution is a greedy solution of E, so E is also greedy. By the induction hypothesis $w(E) = 1$, and by Lemma 10, $w(D) = w(E)$. Consequently, $w(D) = 1$. □

Corollary 12 *If $w(D) < 1$ then D is r-insufficient.*

Proof. By Lemma 10, no pebbling step increases the weight of a rooted distribution, and the weight of a solved rooted distribution is at least 1. It follows that the weight of any r-solvable rooted distribution is at least 1. □

Corollary 13 *For any graph G, $w(G) \geq 1$.* □

Theorem 14 *G is thrifty if and only if $w(G) = 1$.*

Proof. Suppose G is thrifty and D is an r-ceiling distribution on G. Then since G is thrifty, D is greedy. By Lemma 11, D has weight 1. As this is true for any r-ceiling distribution on G, $w(G) = 1$.

Conversely, suppose $w(G) = 1$ and D is an r-ceiling distribution on G. Then D has weight 1. By Lemma 10 and Corollary 13, D must be r-solvable using only greedy pebbling steps. Thus, D is greedy, and G is thrifty. □

Theorem 15 *If G is a thrifty graph with diameter d, then $c_r(G) = 2^d$.*

Proof. Suppose D is an r-ceiling distribution on G. By Lemma 2, $|D| \geq 2^d$. Since G is thrifty, by Lemma 14, $w(D) = 1$. Because every pebble in D must contribute at least $\frac{1}{2^d}$ to $w(D)$, there must be exactly 2^d of them. □

By Theorems 14 and 15, all thrifty graphs achieve the lower bound for weight given in Corollary 12 and the lower bound for r-critical pebbling number given in Lemma 2. Thus, thrifty graphs are in some sense the simplest graphs with respect to r-critical pebbling number. However, although all graphs with weight 1 are thrifty, we prove in Theorem 23 that not all graphs with r-critical pebbling number 2^d are thrifty.

4 Separating Examples

We now consider five specific graphs, which we prove distinguish the classes of greedy graphs, thrifty graphs, and graphs G for which $c_r(G) = 2^d$. The first of these is C_7, the cycle on 7 vertices. The remaining four graphs we call G_1 through G_4, and display them in Figures 7, 8, 9, and 10, respectively. For each graph, we first determine its pebbling number and r-critical pebbling number, and then prove that it has the required properties.

To determine $c_r(C_7)$, it will be useful to have the following lemma. If H is a subgraph of G, D is a rooted distribution on G, and E is a rooted distribution on H, we say that E is *induced* from D if the root of E is the root of D, and $E(a) = D(a)$ for all vertices a of H.

Lemma 16 *If D is a rooted distribution on G, P is a path in G with end vertex r, E is the rooted distribution on P induced from D, and $w(E) > 1$, then D is r-excessive.*

Proof. Suppose that there are no pebbles on r and at most one pebble on every other vertex of P. Then $w(E) < 1$. Assuming that E is not solved, it follows that there exists a vertex a of P for which $E(a) > 1$. Therefore, we may pebble from a towards r. Since this pebbling step is greedy, by Lemma 10, the new rooted distribution E' obtained from this pebbling step still satisfies $w(E') > 1$. We may continue in this way until we reach a solved rooted distribution F. Because $w(F) > 1$, F is r-excessive, so E is r-excessive. As we may use these same pebbling steps on D, D is also r-excessive. □

Theorem 17 *The cycle C_7 is not thrifty, not greedy, and has r-critical pebbling number greater than 2^d.*

Proof. The pebbling number of C_7 is 11 [PSV95]. Let r be any vertex of C_7, and let u and v be the two vertices farthest from r. Then it is easy to verify that the rooted distribution $D(u, v) = (5, 6)$ is not greedy.

Suppose that D is a rooted distribution on C_7 with 11 or more pebbles, and suppose that the number of pebbles in D on each of the six non-root vertices of C_7 are a, b, c, d, e, and f, starting from a vertex adjacent to r and continuing around the cycle. We consider the two paths from r clockwise and counterclockwise around the cycle. If D is r-critical, then by Lemma 16, D must satisfy

$$\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}+\frac{e}{32}+\frac{f}{64}\leq 1 \text{ and}$$
$$\frac{a}{64}+\frac{b}{32}+\frac{c}{16}+\frac{d}{8}+\frac{e}{4}+\frac{f}{2}\leq 1.$$

Adding these inequalities and simplifying yields

$$33a+18b+12c+12d+18e+33f \leq 128$$
$$12(a+b+c+d+e+f)+21a+6b+6e+21f \leq 128.$$

Since $|D| = a+b+c+d+e+f \geq 11$, we have

$$132 = 12\cdot 11 \leq 12(a+b+c+d+e+f)+21a+6b+6e+21f \leq 128,$$

which is a contradiction. Consequently, there is no r-critical distribution on C_7 with 11 or more pebbles, and $c_r(C_7) \leq 10$. As the rooted distribution $D(u,v) = (4,6)$ is r-critical, $c_r(C_7) = 10$. Also, this rooted distribution is not greedy, so C_7 is not thrifty.

Finally, $d(C_7) = 3$, and so $c_r(C_7) = 10 > 8 = 2^d$. $\square$

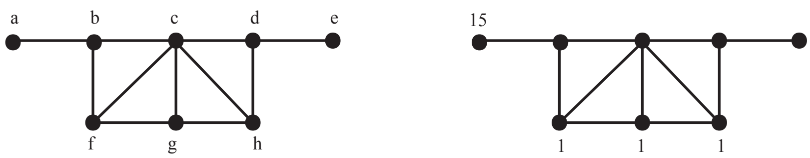

Fig. 7. A labeling and a rooted distribution on the graph G_1.

Lemma 18 $p(G_1) = 18$.

Proof. We follow the labeling system given in Figure 7. Let H be the induced subgraph of G_1 on the vertices b, c, d, f, g, and h. Note that $H \cong F_5$. We will show that any distribution on G_1 with 18 pebbles is solvable.

Let D be a distribution on G_1 with 18 pebbles and assume without loss of generality that $D(a) \geq D(e)$. We may also assume that $D(a)$ is odd because, if $D(a)$ is even, we may add one pebble at a without affecting the solvability of D. There are five cases to consider:

Case 1. $D(a) \geq 17$. In this case, we may pebble to any root from a, since $d(G_1) = 4$.

Case 2. $D(a) = 15$ **and** $r = e$. In this case, we perform the pebbling step $[a, b]$ 7 times, yielding at least 7 pebbles on b, and 10 pebbles total on H. If there is an additional pebble on b, c, or d, or two pebbles on f, g, or h, the resulting distribution is solvable. The only case left is $D(b, f, g, h) = (7, 1, 1, 1)$. Then the rooted distribution is still r-solvable.

Case 3. $D(a) = 13$ **and** $r = e$. We perform the pebbling step $[a, b]$ 6 times, yielding at least 6 pebbles on b, and 11 pebbles total on H. The same arguments as in Case 2 apply here as well.

Case 4. $D(a) \leq 11$ **and** $r = e$. We perform the pebbling step $[a, b]$ as many times as possible, yielding at least 12 pebbles total on H. Since $p(H) = 6$ by Theorem 7, this means that we can pebble two pebbles to d, and one to r.

Case 5. The vertex r is in H. We have 18 pebbles in our distribution. We move as many pebbles as possible from a and e into H. We may leave at most 1 pebble on each of a and e, leaving at least 16 pebbles total. At most half of these may be used moving into H, leaving at least 8 pebbles on H. But $p(H) = 6$ by Theorem 6.

So $p(G_1) \leq 18$. If we remove the pebble on f from the rooted distribution in Figure 7, then this rooted distribution is no longer r-solvable. So $p(G_1) = 18$. □

Lemma 19 $c_r(G_1) = 16$.

Proof. By Lemma 2, $c_r(G_1) \geq 16$. We show that any r-critical distribution on G_1 has at most 16 pebbles. Again, we use the labeling system of Figure 7, and identify H as the induced subgraph of G_1 on the vertices $\{b, c, d, f, g, h\}$.

Case 1. r is a vertex of H.

Let D be a rooted distribution on G_1 with 16 pebbles. We move as many pebbles as possible from a and e into H. We may leave at most 1 pebble on each of a and e, leaving at least 14 pebbles total. At most half of these may be used moving into H, leaving at least 7 pebbles on H. Because $c_r(H) = 5$ by Theorem 8, D is r-excessive. Hence, there are no r-critical distributions on G_1 with r in H. Without loss of generality, there is only one case remaining.

Case 2. $r = e$.

By definition, every r-critical distribution must have a solution S which leaves one pebble on e and no pebble on any other vertex. Let the rooted distributions arrived at after each pebbling step of S be D_{15}, D_{14}, ..., D_2, D_1. We consider the pebbling steps in S in reverse order. The last pebbling step in S must be $[d, e]$, from the rooted distribution D_2 with two pebbles on d and no pebble on any other vertex to the rooted distribution D_1 with one pebble on e.

We color one of the pebbles in D_2 red and the other blue. For each pebbling step before $[d, e]$, we identify the pebble produced from the pebbling step as either red or blue, and color the two input pebbles the same color. Working back through these rooted distributions in this way, we arrive at a coloring of the pebbles in D.

We consider the rooted distributions R and B of only red and blue pebbles, respectively. Note that $\{R, B\}$ is a partition of the distribution D into not necessarily equal sizes. The rooted distributions R and B are each r-critical distributions on $G_1 - e$ with root d, since any non-r-critical solution of one of these rooted distributions would result in a non-r-critical solution of D.

Without loss of generality, a solution of R can begin by pebbling all of the pebbles on a in R to b, obtaining the rooted distribution R'. However, R' is an r-critical distribution on H, so it must have one of the forms given in Lemma 5. This implies that R has at most 8 pebbles, and the only form of R with eight pebbles has all eight pebbles on a. This is also true for B.

Since $|R| \leq 8$ and $|B| \leq 8$, $|D| \leq 16$. So the largest r-critical distributions on G_1 have 16 pebbles, and the only r-ceiling distributions on G_1 have 16 pebbles on a. Consequently, $c_r(G_1) = 16$. □

Note that since $d(G_1) = 4$, $c_r(G_1) = 2^d$.

Theorem 20 *G_1 is thrifty but not greedy.*

Proof. By Lemma 19, the only r-ceiling distributions on G_1 have 16 pebbles at one of the two vertices a and e, and the root at the other vertex. As these rooted distributions are greedy, G_1 is thrifty.

However, the rooted distribution given on G_1 in Figure 7 has $p(G_1)$ pebbles and is not greedy. Thus, G_1 is not greedy. □

Note that since G is thrifty if and only if $w(G) = 1$, G_1 is also an example of a non-greedy graph for which $w(G) = 1$. By Theorem 17, C_7 is not greedy and $w(C_7) > 1$. Thus, although the weight of a graph determines its thrift, it does not determine its greed.

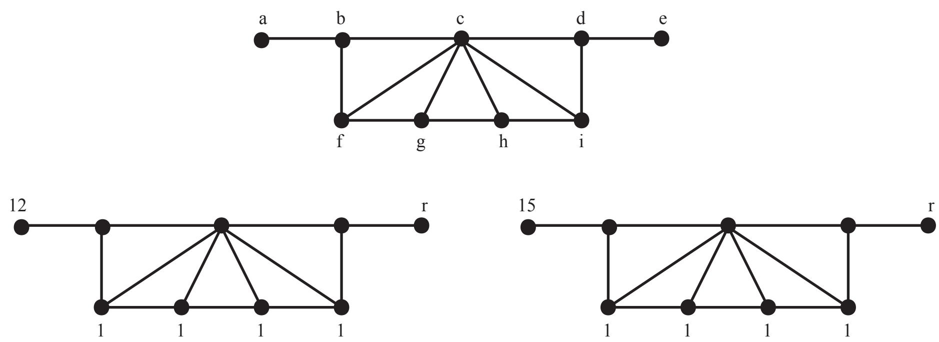

Fig. 8. A labeling and two non-greedy rooted distributions on the graph G_2.

Lemma 21 $p(G_2) = 19$.

Proof. The proof is analogous to the proof of Lemma 18. We use the labeling system of G_2 given in Figure 8. We verify that every rooted distribution on G_2 with 19 pebbles is r-solvable. If a rooted distribution has $r = e$, then we pebble as many pebbles as possible from a to b, and are then able to solve the resulting rooted distribution. If a rooted distribution has r in $\{b, c, d, f, g, h, i\}$ (the analogous subgraph to the subgraph H in Lemma 18), then we pebble as many pebbles as possible from a to b and from e to d, and can then solve the resulting rooted distribution by Theorem 7. So $p(G_2) \leq 19$. If we remove the pebble on f from the second rooted distribution in Figure 8, the resulting rooted distribution is no longer r-solvable. Consequently, $p(G_2) = 19$. □

Lemma 22 $c_r(G_2) = 16$.

Proof. The proof is analogous to the proof of Lemma 19. The only r-critical distributions on G_2 with 16 pebbles have $r = a$ or $r = e$. We assume $r = e$ and

$D(e) = 0$. Every r-critical distribution of this form can be colored red and blue so that the resulting red and blue rooted distributions R and B are r-critical distributions on $G_2 - e$. Each of these rooted distributions can be solved by first pebbling all of the pebbles from a to b, resulting in rooted distributions R' and B', which are r-critical distributions on H. By Lemma 5, these rooted distributions must take one of the five forms given in that lemma. Again, R and B can each have at most eight pebbles, and it follows that $c_r(G_2) = 16$.

Note that given the assumption $r = e$, R must either have $R(a) = 8$ or $R(a, f, g, h, i) = (4, 1, 1, 1, 1)$, and equivalently for B. It follows that the only r-critical distributions on G_2 with 16 pebbles must be a combination of two of these rooted distributions. However, $D(a, f, g, h, i) = (8, 2, 2, 2, 2)$ is not r-critical. Therefore, the only r-critical distributions on G_2 with 16 pebbles are either $D(a) = 16$ or $D(a, f, g, h, i) = (12, 1, 1, 1, 1)$. □

Theorem 23 *G_2 is not thrifty, not greedy, and has r-critical pebbling number 2^d, where d is the diameter of G_2.*

Proof. By Lemma 22, $c_r(G_2) = 16 = 2^d$, because $d(G_2) = 4$. The first rooted distribution in Figure 8 shows a non-greedy r-ceiling distribution on G_2, so G_2 is not thrifty. The second rooted distribution in Figure 8 shows a non-greedy rooted distribution on G_2 with 19 pebbles, which is the pebbling number of G_2 by Lemma 21. □

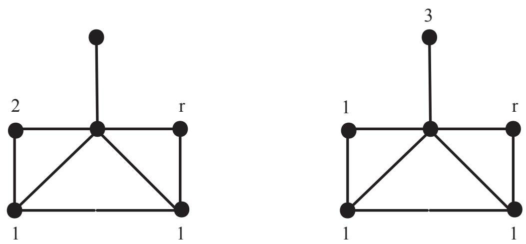

Fig. 9. An r-critical and an r-insufficient distribution on the graph G_3.

Theorem 24 *The graph G_3 shown in Figure 9 is greedy, not thrifty, and has r-critical pebbling number 2^d.*

Proof. The second rooted distribution in Figure 9 is an r-insufficient distribution on G_3 with 6 pebbles. This implies that $p(G_3) \geq 7$.

Let D be a rooted distribution on G_3 with 7 pebbles. We show that D is r-solvable and greedy. If r has a pebble on it, we are done. If every vertex but

r has a pebble on it, then there is one vertex a with more than one pebble. In this case, we can pebble from a to r using a shortest path, so D is r-solvable and greedy.

Now suppose that there is a vertex a other than r with no pebbles on it. That leaves 4 vertices and 7 pebbles. By the pigeonhole principle, D has either one vertex with at least 4 pebbles or two vertices a and b with at least 2 pebbles each. In the first case, the 4 pebbles can be pebbled to r along a shortest path. In the second case, either a or b is adjacent to r or both a and b are adjacent to a neighbor of r. So again D is r-solvable and greedy. Since every rooted distribution on G_3 with 7 pebbles is r-solvable and greedy, $p(G_3) = 7$ and G_3 is greedy.

By Lemma 5, any r-critical distribution D on G_3 with more than 4 pebbles must have at least 2 pebbles on one vertex and at least one pebble on at least 3 other vertices. Call the first vertex a and the other three vertices b, c, and d. For D to be r-critical, the only solution of D must be $([a, b], [b, c], [c, d], [d, r])$. Then the induced subgraph consisting of these five vertices must be a path. But there is no induced P_5 in G_3. So $c_r(G_3) \leq 4$, and because $d(G_3) = 2$, $c_r(G_3) = 4$. In particular, this implies that $c_r(G_3) = 2^d$.

The first rooted distribution D shown in Figure 9 is r-critical and has four pebbles, and so D is an r-ceiling distribution. Therefore, since D is not greedy, G_3 is not thrifty. □

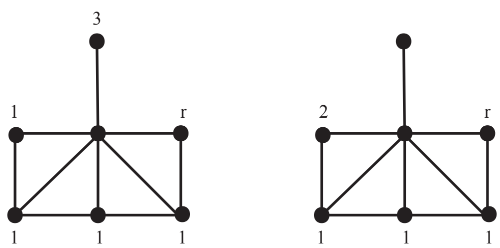

Fig. 10. An r-insufficient and an r-critical distribution on the graph G_4.

Theorem 25 *The graph G_4 shown in Figure 10 is greedy and has r-critical pebbling number greater than 2^d.*

Proof. The proof is analogous to the proof of Theorem 24. The first distribution in Figure 10 is an r-insufficient distribution on G_4 with 7 pebbles. This implies that $p(G_4) \geq 8$.

Let D be a rooted distribution on G_4 with 8 pebbles. As before, we can rule out the case in which D has a pebble on r and the case in which there is at least one pebble on every other vertex. Consider the remaining case, in which there is a vertex a other than r with no pebbles on it. That leaves 5 vertices and 8 pebbles. By the pigeonhole principle, D has either one vertex with at least 4 pebbles or two vertices with at least 2 pebbles each. As before, every rooted distribution on G_4 with 8 pebbles is r-solvable and greedy, so $p(G_4) = 8$ and G_4 is greedy.

Again by Lemma 5, any r-critical distribution D on G_4 with more than 5 pebbles must have 2 pebbles on one vertex and one pebble on at least 4 other vertices. By the same reasoning as in Theorem 24, the induced subgraph consisting of these six vertices must be a path. Since there is no induced P_6 in G_4, $c_r(G_4) \leq 5$. The second rooted distribution D shown in Figure 10 is r-critical and has five pebbles, so $c_r(G_4) = 5$. In particular, $c_r(G_4) > 4 = 2^d$, since $d(G_4) = 2$. □

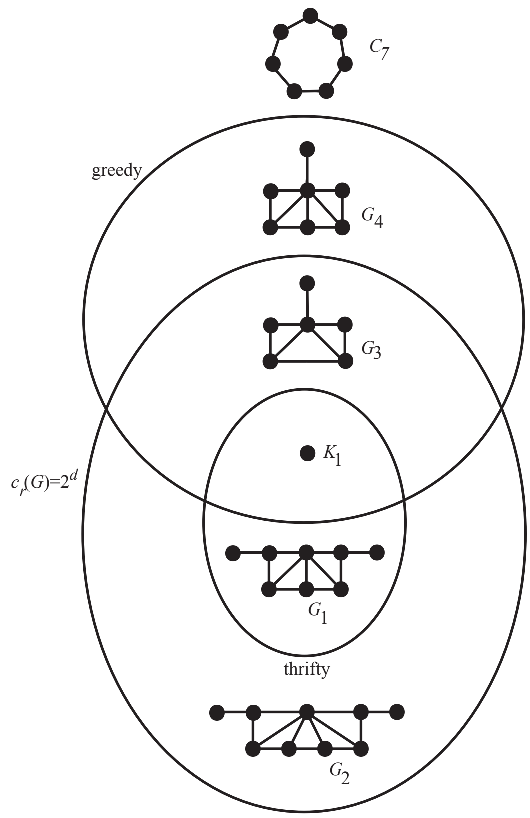

Fig. 11. The Venn diagram of rooted pebbling properties of graphs.

The above results are summarized in the Venn diagram in Figure 11. In each region of this Venn diagram, a graph is shown with the given properties.

Theorem 26 *The fan F_k has weight $\frac{k+1}{4}$.*

Proof. By Theorem 8, $c_r(F_k) = k$. Thus, the weight of F_k is the largest weight of an r-critical distribution D on F_k with k pebbles. No r-critical distribution with more than two pebbles can have more than one pebble in the neighborhood of r. Since $d(F_k) = 2$, this means that at least $k-1$ of the k pebbles in D are distance 2 from r. If all k pebbles in D are distance 2 from r, then $w(D) = \frac{k}{4}$. If exactly $k-1$ pebbles in D are distance 2 from r then $w(D) = \frac{1}{2} + \frac{k-1}{4} = \frac{k+1}{4}$. The first diagram in Figure 6 shows an r-critical distribution with k pebbles and weight $\frac{k+1}{4}$. □

Corollary 27 *There exist graphs with diameter 2 that have arbitrarily large weight.* □

This concludes our discussion of the r-critical pebbling number. We hope to explore the g-critical and u-critical pebbling numbers further in future work.

Acknowledgements

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