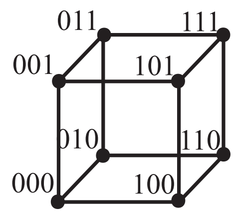
Math 376: Graph Theory
Fall 2023
Homework Assignments

Each assignment is due by midnight on the indicated day. Upload the assignment to your folder in Google Drive. Please work with and discuss these assignments with your professor and classmates, and use appropriate technology such as wolframalpha. Don't copy information from other sources – your written submitted work should be your own.

When I grade your homework problems, I'll comment on the document and put the graded assignments with feedback into a folder called "Returned work" in your Google drive folder. You don't need to respond to my comments, but you're very welcome to talk to me about any of my feedback anytime.

Assignment #1 due Wednesday September 6:

1. Find all the non-isomorphic graphs with 8 or fewer vertices, whose vertices all have degree 3. Draw them all, and explain how you know you found them all.
2. Prove that if G is a simple disconnected graph, then its complement is connected.
3. For an integer $k \geq 1$, the ***cube*** Q_k is the simple graph whose vertices are labeled with all of the sequences of 0s and 1s of length k, with an edge between two vertices x and y if and only if the sequences x and y differ in exactly one number. For example, here's the cube graph Q_3.

   Find with justification the number of vertices, the number of edges, the degrees of all the vertices, the size of the largest clique, and the size of the largest independent set of Q_k in terms of k.
4. For what integers k do there exist k-cycles in the Petersen graph? Justify your answer by showing examples of the values of k that work, and explaining why the other values of k don't work.
5. Prove that the complete bipartite graph $K_{m,n}$ decomposes into two isomorphic subgraphs if and only if m and n are not both odd.

Assignment #2 due Friday September 22:

1. (a) For which k is the cube graph Q_k Eulerian? Justify your answer.

   (b) For which k is the cube graph Q_k Hamiltonian? Justify your answer.
2. Suppose G is a connected simple graph with no clique of size 3, and no P_4 (path with four vertices) as an induced subgraph. Prove that G is a complete bipartite graph.
3. Suppose G is a connected simple graph, and P and Q are paths of maximum length in G, i.e., no path in G has length larger than P or Q. Prove that P and Q have a common vertex.

4. In a connected graph G, the ***distance*** $d(v, w)$ between two vertices v and w is the smallest number of edges in any path between them. The ***diameter*** of G is the largest distance between any two vertices in G. The ***eccentricity*** of a vertex v of G is the largest distance from v to any other vertex in G. The ***radius*** of G is the smallest eccentricity of any vertex in G.

   Find all possible diameters of graphs with radius 3. For each diameter that is possible, draw an example. For each diameter that's not possible, justify why it's not possible.

5. A graph is ***claw-free*** if it has no induced subgraph isomorphic to $K_{1,3}$. Write a Python program to input a simple graph and output whether it's claw-free. Put your program in a Google Colab file in your folder.

Assignment #3 due Wednesday October 4:

1. How many non-isomorphic trees are there with n vertices and diameter 3? Find a formula, and prove that your formula works.
2. Prove that if G is a tree with a vertex of degree k, then G has at least k leaves.
3. (a) Suppose $d_1, d_2, \ldots, d_n$ are positive integers with $n \geq 2$ and $\sum d_i = 2n - 2$. Show that there is a tree with vertex degrees $d_1, d_2, \ldots, d_n$.

   (b) For $n = 7$, write a Python program that inputs seven integers of this form and outputs a tree with these degrees. Include a command to draw the tree. You can assume the integers are sorted from biggest to smallest. Put your program in a Google Colab file in your folder.
4. Prove that a graph G is connected if and only if it contains two vertices x and y such that $G - x$ and $G - y$ are both connected.
5. For all positive integers n and k with $n > k$, find with proof the simple graphs with n vertices for which every induced subgraph with k vertices is a tree.

Assignment #4 due Wednesday, October 18:

1. The ***line graph*** $L(G)$ of a graph G is the graph with a vertex for each edge of G, with two vertices of $L(G)$ adjacent if and only if their corresponding edges share a vertex. The ***cartesian product*** $G \square H$ of two graphs G and H is the graph with vertex set $\{(x, y) \mid x \in G, y \in H\}$, i.e. the set of ordered pairs of vertices of G and vertices of H, and with an edge between two vertices (a, b) and (c, d) if either $a = c$ and $b \leftrightarrow d$ or $a \leftrightarrow c$ and $b = d$.

   Prove that $L(K_{m,n}) \cong K_m \square K_n$.
2. A ***minimax spanning tree*** T of a connected weighted graph G is the spanning tree that minimizes the largest weight of an edge of T (as opposed to the minimum weight spanning tree, which minimizes the sum of the weights of T). For a given connected weighted graph G, is a minimum weight spanning tree always minimax? Is a minimax spanning tree always minimum weight? If yes, give a proof. If no, give a counterexample.
3. For positive integers k and j, find the maximum number of edges in a simple bipartite graph that has no matching with k edges and no star with j edges.
4. For each integer $k > 1$, find a k-regular graph (all vertices have degree k) with no perfect matching.

5. Write a Python program that finds a maximal (not necessarily maximum) matching of a graph. Output the list of edges in the matching.

Assignment #5 due Friday, November 3:

1. Find two non-isomorphic tournaments with the same list of outdegrees.
2. Let G be an acyclic tournament with n vertices. Prove that the vertices of G can be labeled with the numbers 1 through n such that for all vertices x and y in G, there's an edge from vertex x to vertex y if and only if the label on x is less than the label on y.
3. A ***homomorphism*** between two simple graphs G and H is a surjection $\varphi : V(G) \to V(H)$ such that $vw \in E(G)$ implies $\varphi(v)\varphi(w) \in E(H)$. (Note the two differences from the definition of an isomorphism.) Prove that G is k-colorable if and only if there exists a homomorphism from G to K_k.
4. For each vertex v of a graph G, let $L(v)$ denote a list of available colors at v. A ***list coloring*** of G is a proper coloring of G in which every vertex is colored with a color from its list. G is ***list*** k***-colorable*** if every assignment of k-element lists to the vertices of G permits a proper list coloring. (Note that these lists may contain more than k colors total, even though each of them has only k colors.) The ***list chromatic number*** of G is the smallest k for which G is list k-colorable.

   Recall the Cartesian product of graphs from assignment 4. Prove that the list chromatic number of $P_m \Box P_n$ is 3 for all positive integers m and n with $m \geq 2$ and $n \geq 3$.
5. Using the Graph Atlas in WISE, determine what percentage of 7-vertex graphs are 3-colorable using the Python `greedy_color` command.
   Extra credit: Find any graph for which `greedy_color` doesn't find an optimal coloring.

Assignment #6 due Wednesday, November 15:

1. (a) Let G and H be k-critical graphs sharing only vertex x, where $x \leftrightarrow y$ is an edge in G and $x \leftrightarrow z$ is an edge in H. Prove that the graph obtained from G and H by deleting the edges $x \leftrightarrow y$ and $x \leftrightarrow z$ and adding the edge $y \leftrightarrow z$ is also k-critical.

   (b) For all $n \geq 6$, construct a 4-critical graph with n vertices.
2. Determine with justification which of the graphs $(C_n)^{\mathcal{C}}$ (the complements of cycle graphs) are planar, for $n \geq 3$.
3. Recall that a region of the plane F is ***convex*** if, for any two points x and y in F, the line segment xy is also in F. Does every planar graph have a planar embedding in which all its non-infinite faces are convex? Justify your answer.
4. Let H be a graph with maximum degree 3. Prove that any graph G contains a subdivision of H if and only if G contains a subgraph contractible to H.
5. Write a Python program to input a non-planar graph G and output a maximal planar subgraph H of G, with a planar drawing of H.

Assignment #7 due Monday, December 6:

1. Let G be a connected plane graph, and let S be a set of edges of G. Prove that S is a spanning tree of G if and only if the duals of the edges of G not in S form a spanning tree of the dual of G.

2. A graph G has ***thickness*** 2 if G is not planar, but G is the union of 2 planar graphs. Prove that the cube Q_4 has thickness 2.

3. An ***interval representation*** of a graph G is a set of closed intervals S on the real line, and a bijection between the vertices of G and the intervals in S, such that two vertices are adjacent in G if and only if their corresponding intervals intersect. A graph is an ***interval graph*** if there exists an interval representation of G.

   Recall that a ***caterpillar*** is a tree which is the union of a path (the *spine*) and some additional leaves that have an edge to a vertex on the spine (the *feet*).

   Prove that a tree G is an interval graph if and only if G is a caterpillar.

4. A ***graceful*** labeling of a graph G with m edges labels the vertices of G with numbers between 0 and m inclusive, and the edges of G with the difference of the labels of their vertices, such that every vertex gets a different number, and every edge gets a different number (so the edges are labeled with exactly the numbers 1 through m once each). Show that every caterpillar with at most two feet has a graceful labeling.

5. Write a Python program to input a graph G and output whether G has either a clique of size 5 or an independent set of size 5. Using the Python command `fast_gnp_random_graph`, run your program on some random graphs with 43 vertices. Based on how long your program takes on a few graphs, roughly estimate how long it would take to check all graphs with 43 vertices (and thus determine whether the Ramsey number $R(5,5) = 43$).