Theorem 1 (Kuratowski's Theorem, 1930). *A graph is planar if and only if it doesn't contain a subdivision of K_5 or $K_{3,3}$.*

Theorem 2 (Wagner, 1937). *A graph is planar if and only if it doesn't contain a K_5 minor or a $K_{3,3}$ minor.*

Definitions:

- A ***Kuratowski subgraph*** of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$.
- A ***minimal non-planar graph*** is a non-planar graph G for which every proper subgraph of G is planar.
- A ***vertex cut*** of G is a set of vertices whose removal disconnects G. Note that a cut vertex is a vertex cut with one vertex.
- Given a vertex cut S of G, an S***-lobe*** of G is the induced subgraph of G whose vertices are S and a component of $G - S$.
- A ***convex embedding*** of a planar graph G is a planar embedding of G for which every face is convex.

Proof of Kuratowski's Theorem.

Part I. If G has a Kuratowski subgraph then G is not planar.

Part II. If G is not planar then G has a Kuratowski subgraph.
We prove this by minimal counterexample. Suppose G is a minimal non-planar graph and has no Kuratowski subgraph.

1. G is 3-connected.

   (a) Given any face F of a planar embedding of G, we can embed G in the plane with F as the outside face.

   (b) Every minimal non-planar graph is 2-connected.

   (c) If $S = \{x, y\}$ is a vertex cut of G, then the graph obtained by adding the edge xy to an S-lobe of G is not planar.

   (d) Since G is minimal non-planar, G has no vertex cut $\{x, y\}$, so G is 3-connected.

2. Every 3-connected graph with no Kuratowski subgraph is planar.

   We prove a stronger statement by induction: Every 3-connected graph with no Kuratowski subgraph has a convex embedding.

   (a) **Base case.** K_4 works.

   (b) **Induction step.** Assume G is 3-connected, has no Kuratowski subgraph, and has at least 5 vertices.

      i. G has an edge e such that $G' = G \cdot e$ is 3-connected. (By contradiction)

         A. Suppose $G \cdot e$ is not 3-connected for any edge e. Then for any two adjacent vertices x and y of G, there is a third vertex z of G such that $G - \{x, y, z\}$ is disconnected.

         B. Among all such triples of vertices x, y, and z, choose x, y, and z so that $G - \{x, y, z\}$ has a component H with the most vertices. Let H' be another component of $G - \{x, y, z\}$. All three vertices x, y, and z have edges to both H and H'.

         C. Let $u \in H'$ be adjacent to z. There is a vertex v in G such that $G - \{u, v, z\}$ is disconnected.

         D. Let J be the induced subgraph of G with vertices $V(H) \cup \{x, y\}$. J may or may not contain v, but either way $J - v$ is connected.

         E. $J - v$ is contained in a component of $G - \{u, v, z\}$ with more vertices than H, contradicting the choice of H.

      ii. G' doesn't have a Kuratowski subgraph. (By contrapositive)

         A. Suppose G' has a Kuratowski subgraph H. Let w be the vertex in G' obtained by contracting e. If $w \notin V(H)$ then G has a Kuratowski subgraph.

         B. If $w \in V(H)$ and w has degree 2 in H then G has a Kuratowski subgraph.

         C. If $w \in V(H)$ and at most one edge incident to w in H is incident to x in G, then G has a Kuratowski subgraph.

         D. Suppose $w \in V(H)$ at least two edges incident to w in H are incident to x in G, and at least two edges incident to w in H are incident to y in G. Then H has a subdivision of K_5.

         E. In this case, G has a subdivision of $K_{3,3}$.

      iii. G' has a convex embedding.

      iv. G has a convex embedding.

A. Let w be the vertex in G' obtained by contracting e. In a convex embedding of G', let G'' be the graph obtained by deleting the edges incident to w. The boundary of the face of G'' containing w is a cycle C of G' and of G.
B. Each edge from w to C in G' must be an edge from x or y, or both, to C in G.
C. If x and y share 3 neighbors of C, then G has a subdivision of K_5, a contradiction.
D. If x and y alternate neighbors a, b, c, d on C, then G has a subdivision of $K_{3,3}$, a contradiction.
E. So x and y share at most two neighbors on C, and all the neighbors of y appear between two consecutive vertices of x.
F. We can fit x and y in the convex embedding of G' to make a convex embedding of G.

□