Orthogonal complements

If S is a set of vectors in $\mathbb{R}^n$, the ***orthogonal complement*** of S, denoted $S^\perp$ (S *perp*) is the set of vectors orthogonal to all vectors in S.

What are the orthogonal complements of these sets?

1. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ in $\mathbb{R}^2$
2. The line $y = 2x$ in $\mathbb{R}^2$
3. The plane $z = 2x + 3y$ in $\mathbb{R}^3$

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The ***row space*** $\text{row}(A)$ of a matrix A is the span of the rows of A.

Properties of orthogonal complements:

Suppose S is a nonempty subset of $\mathbb{R}^n$, V is a subspace of $\mathbb{R}^n$, and A is an $m \times n$ matrix.

1. $S^\perp$ is a subspace of $\mathbb{R}^n$.
2. $V \cap V^\perp = \vec{0}$. (V and $V^\perp$ have only $\vec{0}$ in common)
3. $S^\perp = (\text{span}(S))^\perp$.
4. $(V^\perp)^\perp = V$.
5. $\text{null}(A) = (\text{row}(A))^\perp$
6. $\text{null}(A^T) = (\text{col}(A))^\perp$

Theorem (The Rank Theorem)

♡♡♡ $\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{row}(A)).$ ♡♡♡

Note $\text{col}(A)$ is a subspace of $\mathbb{R}^n$ and $\text{row}(A)$ is a subspace of $\mathbb{R}^m$.

The ***nullity*** of A is the dimension of $\text{null}(A)$.

Theorem (Rank-Nullity Theorem AKA Dimension Theorem for Matrices)

♢♢♢ *If A is an $m \times n$ matrix,* $\text{rank}(A) + \text{nullity}(A) = n.$ ♢♢♢

Theorem (Dimension Theorem for Subspaces)

♣♣♣ *If V is a subspace of $\mathbb{R}^n$,* $\dim(V) + \dim(V^\perp) = n.$ ♣♣♣