Coordinates with respect to a basis

Given a vector $\vec{w}$ in $\mathbb{R}^n$ and a basis $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of $\mathbb{R}^n$, the ***coordinates*** of $\vec{w}$ with respect to B (or the ***B-coordinates*** of $\vec{w}$) are

$$[\vec{w}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \text{ where } \vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

Five increasingly tricky problems:

1. Given $\vec{w}$ in B-coordinates, find $\vec{w}$ in standard coordinates.
2. Given $\vec{w}$ in standard coordinates, find $\vec{w}$ in B-coordinates.
3. Given $\vec{w}$ in B_1-coordinates, find $\vec{w}$ in B_2-coordinates.
4. Given a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, find the matrix for T which inputs and outputs in B-coordinates.
5. Given the matrix for T in B_1-coordinates, find the matrix for T in B_2-coordinates.

Solutions

1. Given $\vec{w}$ in B-coordinates, find $\vec{w}$ in standard coordinates.
   Plug into $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ and simplify.

Solutions

1. Given $\vec{w}$ in B-coordinates, find $\vec{w}$ in standard coordinates.
   Plug into $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ and simplify.
2. Given $\vec{w}$ in standard coordinates, find $\vec{w}$ in B-coordinates.
   Solve $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ for $a_1, \ldots, a_n$.

Solutions

1. Given $\vec{w}$ in B-coordinates, find $\vec{w}$ in standard coordinates.
   Plug into $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ and simplify.

2. Given $\vec{w}$ in standard coordinates, find $\vec{w}$ in B-coordinates.
   Solve $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ for $a_1, \ldots, a_n$.

3. Given $\vec{w}$ in B_1-coordinates, find $\vec{w}$ in B_2-coordinates.
   Find B_2-coordinates for the vectors in B_1, and plug into $\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$.

   The general formula is $[\vec{w}]_{B_2} = \left[[\vec{v}_1]_{B_2}[\vec{v}_2]_{B_2} \cdots [\vec{v}_n]_{B_2}\right] [\vec{w}]_{B_1}$.
   The matrix $\left[[\vec{v}_1]_{B_2}[\vec{v}_2]_{B_2} \cdots [\vec{v}_n]_{B_2}\right]$ is the ***transition matrix*** or ***change of coordinates matrix*** $P_{B_1 \to B_2}$ from B_1 to B_2.

   We can find $P_{B_1 \to B_2}$ by reducing $[B_2|B_1]$ to obtain $[I_n|P_{B_1 \to B_2}]$.

Theorem

$T(\vec{x}) = [\vec{x}]_B$ is an invertible linear operator, and $(P_{B_1 \to B_2})^{-1} = P_{B_2 \to B_1}$.

4. Given a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, find the matrix for T which inputs and outputs in B-coordinates.
   In other words, find the matrix $[T]_B$ which inputs $[\vec{x}]_B$ and outputs $[T(\vec{x})]_B$.

$$[T]_B = [[T(\vec{v}_1)]_B [T(\vec{v}_2)]_B \cdots [T(\vec{v}_n)]_B]$$

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5. Given the matrix for T in B_1-coordinates, find the matrix for T in B_2-coordinates.
   In other words, given $[T]_{B_1}$ find $[T]_{B_2}$, the matrix that inputs $[\vec{x}]_{B_2}$ and outputs $[T(\vec{x})]_{B_2}$.

   We know $P_{B_1 \to B_2}[\vec{x}]_{B_1} = [\vec{x}]_{B_2}$.

   So we convert $[\vec{x}]_{B_2}$ to $[\vec{x}]_{B_1}$, then to $[T(\vec{x})]_{B_1}$, then to $[T(\vec{x})]_{B_2}$:

$$[T]_{B_2} = P_{B_1 \to B_2}[T]_{B_1} P_{B_2 \to B_1}$$
$$[T]_{B_2} = P_{B_1 \to B_2}[T]_{B_1} (P_{B_1 \to B_2})^{-1}$$