Four different ways of computing the matrix product AB:

1. The entry in the ith row and jth column of AB, denoted $(AB)_{ij}$, is the dot product of the ith row of A with the jth column of B.
2. $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$, where the vectors $\vec{a}_1, \ldots, \vec{a}_n$ are the columns of A.
3. $AB = \left[A\vec{b}_1\, A\vec{b}_2 \cdots A\vec{b}_n\right]$, where the vectors $\vec{b}_1, \ldots, \vec{b}_n$ are the columns of B.
4. $AB = \begin{bmatrix} \vec{r}_1 B \\ \vec{r}_2 B \\ \vdots \\ \vec{r}_m B \end{bmatrix}$ where the vectors $\vec{r}_1, \ldots, \vec{r}_n$ are the rows of A.

Some properties of matrix products

1. To multiply AB, we need the number of columns of A equal to the number of rows of B.
2. Most of the time $AB \neq BA$. (Matrix multiplication is not commutative.)
3. But $(AB)C = A(BC)$ as long as you keep them in the same order. (Matrix multiplication is associative.)
4. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$
5. $A\vec{x}$ works as a matrix product if $\vec{x}$ is a column vector.
6. Multiplying by a scalar is commutative: $A(c\vec{x}) = c(A\vec{x}) = (cA)\vec{x}$.
7. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ and $A(B + C) = AB + AC$.
(Matrix multiplication is distributive over matrix addition.)

True or False?

1. If AB and BA are both defined, then A and B are both square matrices.
2. If B has a column of zeros, then so does AB.
3. If A has a column of zeros, then so does AB.
4. If A has two rows repeated, then so does AB.
5. If A has two columns repeated, then so does AB.
6. If $AB = 0$ then $A = 0$ or $B = 0$.
7. If $AC = BC$ then $A = B$.

Special matrices and other matrix operations:

- A ***zero matrix*** $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$
- An ***identity matrix*** $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
- The ***transpose*** of A, A^T is formed by swapping rows and columns of A, equivalently, reflecting A across its main diagonal, equivalently, $\left(A^T\right)_{ij} = (A)_{ji}$.
- The ***trace*** of A, $\mathtt{tr}(A)$ is the sum of the entries on the main diagonal of A.
- An $n \times n$ matrix A is ***invertible*** or ***nonsingular*** if there is a matrix A^{-1} (the ***inverse*** of A) such that $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

Properties of Transposes and Inverses

1. $(AB)^T = B^T A^T$.
2. If A is invertible, then A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$.
3. If A is an invertible matrix, then A^{-1} is invertible, and $(A^{-1})^{-1} = A$.
4. If A and B are invertible $n \times n$ matrices, then AB is invertible, and
$$(AB)^{-1} = B^{-1}A^{-1}.$$
5. If A is an invertible $n \times n$ matrix and k is a positive integer, then A^k is invertible, and
$$(A^k)^{-1} = (A^{-1})^k.$$
6. If A is an invertible matrix, then the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.