Linear independence

- Recall that the vector $\vec{w}$ is a ***linear combination*** of the vectors $\vec{v}_1, \ldots, \vec{v}_k$ if there exist scalars $c_1, \ldots, c_k$ such that
$$\vec{w} = c_1 \vec{v}_1 + \ldots + c_k \vec{v}_k.$$
- A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is ***linearly independent*** if the only linear combination of the vectors that gives $\vec{0}$ is $0\vec{v}_1 + \ldots + 0\vec{v}_k$. More formally, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent if $c_1 \vec{v}_1 + \ldots + c_k \vec{v}_k = \vec{0}$ implies $c_1 = c_2 = \cdots = 0$.

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Which sets of vectors are linearly independent?

1. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$
2. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
3. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
4. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix} \right\}$
5. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

Theorems about linear independence

1. The columns of a matrix A are linearly independent if and only if $A\vec{x} = \vec{0}$ has exactly one solution, namely, $\vec{x} = \vec{0}$.
2. A set consisting of a single vector is linearly independent if and only if the vector is not the zero vector.
3. A set of two vectors is linearly independent if and only if the vectors are both not zero, and not multiples of each other.
4. Any set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
5. Any set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ which contains the zero vector is linearly dependent.
6. If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ is linearly independent, then $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_{p-1}\}$ is also linearly independent.
7. If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_{p-1}\}$ is linearly dependent, then $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ is also linearly dependent.

Subspaces

A nonempty set W of vectors in $\mathbb{R}^n$ is called a ***subspace*** of $\mathbb{R}^n$ if

1. If $\vec{u}$ is in W then $c\vec{u}$ is in W for any real number c.
   (closed under scalar multiplication)
2. If $\vec{u}$ is in W and $\vec{v}$ is in W then $\vec{u} + \vec{v}$ is in W.
   (closed under vector addition)

Equivalently, every linear combination of vectors in W is also in W.
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Which sets are subspaces?

1. The x-axis in $\mathbb{R}^2$.
2. The line $y = 2x + 3$ in $\mathbb{R}^2$.
3. The first quadrant in $\mathbb{R}^2$.
4. The solid ball $x^2 + y^2 + z^2 \leq 1$ in $\mathbb{R}^3$.
5. The plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

Theorems about subspaces

The set of all linear combinations of the vectors $\vec{v}_1, \ldots, \vec{v}_k$ is called the ***span*** of $\vec{v}_1, \ldots, \vec{v}_k$ and is denoted $\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$.

Theorems about subspaces

1. A subset of $\mathbb{R}^n$ is a subspace if and only if it's the span of some finite set of vectors $\vec{v}_1, \ldots, \vec{v}_k$.
2. $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.
3. $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
4. A subspace of $\mathbb{R}^2$ is either the origin, a line through the origin, or all of $\mathbb{R}^2$.
5. A subspace of $\mathbb{R}^3$ is either the origin, a line through the origin, a plane through the origin, or all of $\mathbb{R}^3$.
6. If A is an $m \times n$ matrix, the set of solutions to the homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.