A ***linear transformation*** is a function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies

1. $T(c\vec{u}) = cT(\vec{u})$ for all vectors $\vec{u}$ in $\mathbb{R}^n$ and all real numbers c. (Homogeneity)
2. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$. (Additivity)

If $m = n$ then T is also called a ***linear operator*** on $\mathbb{R}^n$.

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A ***matrix transformation*** is a function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$ for some $m \times n$ matrix A.

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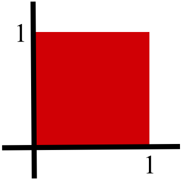
Corollary. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(\vec{x}) = A\vec{x}$, where $A = [T(\vec{e}_1)T(\vec{e}_2)\cdots T(\vec{e}_n)]$.

What do these linear transformations do to this square?



1. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
2. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
3. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
4. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
5. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
6. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
7. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Recall:

- The **inverse image** $T^{-1}(\vec{b})$ under the function T is the set of vectors $\vec{x}$ for which $T(\vec{x}) = \vec{b}$.
- The **range** of the function T is the set of outputs of T.

Properties of linear transformations:

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation defined by $T(\vec{x}) = A\vec{x}$, then

1. $T(\vec{0}) = \vec{0}$.
2. $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_pT(\vec{v}_p)$. (Linear transformations preserve linear combinations.)
3. The range of T is the span of the columns of A.
4. The vector $\vec{b}$ is in the range of T if and only if $A\vec{x} = \vec{b}$ is consistent.
5. The set of solutions to $A\vec{x} = \vec{b}$ is the inverse image $T^{-1}(\vec{b})$.
6. The range of T is a subspace of $\mathbb{R}^m$.
7. The inverse image $T^{-1}(\vec{0})$ is a subspace of $\mathbb{R}^n$.

Orthogonal transformations

Theorem

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation defined by $T(\vec{x}) = A\vec{x}$. The following are equivalent.

1. $||T(\vec{x})|| = ||\vec{x}||$ *for all* $\vec{x} \in \mathbb{R}^n$ *(T is* ***orthogonal****).*
2. $||A\vec{x}|| = ||\vec{x}||$ *for all* $\vec{x} \in \mathbb{R}^n$
3. $A^{-1} = A^T$ *(A is* ***orthogonal****).*
4. $A^T A = I$.
5. $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ *for all* $\vec{x}$ *and* $\vec{y}$ *in* $\mathbb{R}^n$.
6. *The columns of A are orthonormal (orthogonal and length 1).*
7. *The rows of A are orthonormal.*

Orthogonal transformations preserve lengths and angles, so they correspond to rigid motions in space.