Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix A.

Vocabulary relating to T:

- The **range** $\mathrm{ran}(T)$ is the set of outputs of T.
- The **kernel** $\ker(T)$ is the inverse image $T^{-1}(\vec{0})$ (the set of input vectors $\vec{x}$ for which $T(\vec{x}) = \vec{0}$).
- T is **onto** or **surjective** if $\mathrm{ran}(T) = \mathbb{R}^m$.
- T is **one-to-one** or **injective** if different inputs $\vec{x}_1 \neq \vec{x}_2$ produce different outputs $T(\vec{x}_1) \neq T(\vec{x}_2)$.

Vocabulary relating to A:

- The **column space** $\mathrm{col}(A)$ is the span of the columns of A.
- The **null space** $\mathrm{null}(A)$ is the set of solutions of $A\vec{x} = \vec{0}$.

Theorem

T is injective if and only if $\ker(T) = \{\vec{0}\}$.

Theorem

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by $T(\vec{x}) = A\vec{x}$.

1. *T is surjective if and only if the columns of A span $\mathbb{R}^m$.*
2. *T is injective if and only if the columns of A are linearly independent.*

Theorem

If $m = n$, T is injective if and only if T is surjective.

Theorem

n vectors in $\mathbb{R}^n$ are linearly independent if and only if they span $\mathbb{R}^n$.

A **basis** of $\mathbb{R}^n$ is a set of vectors that are linearly independent and span $\mathbb{R}^n$.

Updated Amazing Awesome Unifying Invertible Matrix Theorem

Theorem. Suppose A is an $n \times n$ matrix. The following are equivalent.

1. A is invertible.
2. A is the product of elementary matrices.
3. The reduced row echelon form of A is I_n.
4. A has n pivot variables in its reduced row echelon form. (i.e. $\mathrm{rank}(A) = n$).
5. $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$. (i.e. $\mathrm{null}(A) = \vec{0}$.)
6. $A\vec{x} = \vec{b}$ has at least one solution for all $\vec{b}$ in $\mathbb{R}^n$. (i.e. $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^n$.)
7. $A\vec{x} = \vec{b}$ has at most one solution for all $\vec{b}$ in $\mathbb{R}^n$.
8. $A\vec{x} = \vec{b}$ has exactly one solution for all $\vec{b}$ in $\mathbb{R}^n$.
9. There is an $n \times n$ matrix C such that $CA = I_n$.
10. There is an $n \times n$ matrix D such that $AD = I_n$.
11. A^T is invertible.
12. The columns of A are linearly independent.
13. The columns of A span $\mathbb{R}^n$. (i.e. $\mathrm{col}(A) = \mathbb{R}^n$.)
14. The columns of A form a basis of $\mathbb{R}^n$.
15. The linear transformation $T(\vec{x}) = A\vec{x}$ is injective.
16. The linear transformation $T(\vec{x}) = A\vec{x}$ has kernel $\{\vec{0}\}$ (i.e. $T^{-1}(\vec{0}) = \{\vec{0}\}$).
17. The linear transformation $T(\vec{x}) = A\vec{x}$ is surjective. (i.e. $\mathrm{ran}(T) = \mathbb{R}^n$.)
18. The linear transformation $T(\vec{x}) = A\vec{x}$ is a bijection (both injective and surjective).
19. The linear transformation $T(\vec{x}) = A\vec{x}$ is invertible.
20. The rows of A are linearly independent.
21. The rows of A span $\mathbb{R}^n$.
22. The rows of A form a basis of $\mathbb{R}^n$.

Theorem (Linear transformation composition = matrix multiplication)

If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear transformations given by $T_1(\vec{x}) = A\vec{x}$ and $T_2(\vec{x}) = B\vec{x}$ then $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation given by $T_2 \circ T_1(\vec{x}) = BA\vec{x}$.

Theorem (Linear transformation inverse = matrix inverse)

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation given by $T(\vec{x}) = A\vec{x}$ then $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation given by $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

Theorem

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation.

1. *If L is a line in $\mathbb{R}^n$ then $T(L)$ is a line in $\mathbb{R}^n$.* ***(T preserves lines.)***
2. *If L_1 and L_2 are parallel lines then $T(L_1)$ and $T(L_2)$ are parallel.* ***(T preserves parallel lines.)***
3. *If the point $\vec{x}$ lies on the line L, then the point $T(\vec{x})$ lies on the line $T(L)$.* ***(T preserves incidence.)***
4. *If three points $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ lie on the same line, then $T(\vec{x}_1)$, $T(\vec{x}_2)$, and $T(\vec{x}_3)$ lie on the same line.* ***(T preserves collinearity.)***
5. *If S is the set of points between $\vec{x}_1$ and $\vec{x}_2$ on the line L, then $T(S)$ is the set of points between $T(\vec{x}_1)$ and $T(\vec{x}_2)$ on the line $T(L)$.* ***(T preserves betweenness.)***