

Finite Prime Distance Graphs and 2-Odd Graphs

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Abstract

A graph G is a prime distance graph (respectively, a 2-odd graph) if its vertices can be labeled with distinct integers such that for any two adjacent vertices, the difference of their labels is prime (either 2 or odd). We prove that trees, cycles, and bipartite graphs are prime distance graphs, and that Dutch windmill graphs and paper mill graphs are prime distance graphs if and only if the Twin Prime Conjecture and dePolignac's Conjecture are true, respectively. We give a characterization of 2-odd graphs in terms of edge colorings, and we use this characterization to determine which circulant graphs of the form $\text{Circ}(n, \{1, k\})$ are 2-odd and to prove results on circulant prime distance graphs.

Keywords: distance graphs, prime distance graphs, difference graphs

1. Introduction

Prime distance graphs were introduced by Eggleton, Erdős, and Skilton in 1985 [4, 5]. For any set D of positive integers, they defined the *distance graph* $\mathbb{Z}(D)$ as the graph with vertex set \mathbb{Z} and an edge between integers x and y if and only if $|x - y| \in D$. The *prime distance graph* $\mathbb{Z}(P)$ is the distance graph with $D = P$, the set of all primes. They proved that the chromatic number $\chi(\mathbb{Z}(P)) = 4$. Research in prime distance graphs has since focused on the chromatic number of $\mathbb{Z}(D)$ where D is a non-empty proper subset of P [7, 6, 17, 19]. Note that these graphs are all infinite (non-induced) subgraphs of $\mathbb{Z}(P)$. In this paper we investigate finite subgraphs of $\mathbb{Z}(P)$.

Specifically, we say that a graph G is a **prime distance graph** if there exists a one-to-one labeling of its vertices $L : V(G) \rightarrow \mathbb{Z}$ such that for any

two adjacent vertices u and v , the integer $|L(u) - L(v)|$ is prime. We also define $L(uv) = |L(u) - L(v)|$. We call L a **prime distance labeling** of G , so G is a prime distance graph if and only if there exists a prime distance labeling of G . We sometimes denote a vertex with label i by (i) . Note that in a prime distance labeling, the labels on the vertices of G must be distinct, but the labels on the edges need not be. Also note that by our definition, $L(uv)$ may still be prime if uv is not an edge of G .

We say that G is **2-odd** if for any two adjacent vertices u and v , $|L(u) - L(v)|$ is either odd or exactly 2, in which case L is a **2-odd labeling** of G . Note that prime distance graphs are trivially 2-odd. Corollary 22 below shows that not every 2-odd graph is a prime distance graph.

Example 1. The path P_n is 2-odd and prime-distance for each n . Figure 1 shows a prime distance (and therefore 2-odd) labeling of P_n .

□

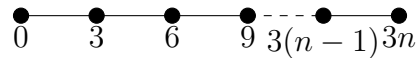


Figure 1: A prime distance labeling of P_n .

2. Connections to Well-known Statements in Number Theory

Surprisingly, the existence of prime distance labelings of some infinite families of graphs is closely related to several well-known statements in Number Theory. In this section we show that all bipartite graphs are prime distance graphs using the Green-Tao Theorem; we give three separate proofs that all cycle graphs are prime distance graphs using the Goldbach Conjecture, Vinogradov's Theorem, and Ramaré's Theorem, respectively; and we show that Dutch windmill graphs and paper mill graphs are prime distance graphs if and only if the Twin Prime Conjecture and dePolignac's Conjecture are true, respectively.

Recall that an **arithmetic progression** of positive integers is a sequence of positive integers such that the differences between successive terms of the sequence is a constant.

The Green-Tao Theorem. [10] *For any positive integer k , there exists a prime arithmetic progression of length k .*

Theorem 1. *Every bipartite graph is a prime distance graph.*

Proof. We show that the graph $K_{r,s}$ is a prime distance graph. Since every subgraph of a prime distance graph is also a prime distance graph, this proves that every bipartite graph is a prime distance graph. By the Green-Tao Theorem, there is an arithmetic sequence of $r + s - 1$ primes $p - (r - 1)k, p - (r - 2)k, \dots, p - k, p, p + k, \dots, p + (s - 2)k, p + (s - 1)k$. Let A and B be the partite sets of G with $|A| = r$ and $|B| = s$. Label the members of B with the labels $p, p + k, \dots, p + (s - 1)k$ and the members of A with the labels $0, k, 2k, \dots, (r - 1)k$. Then differences between members of A and members of B are all of the form $p + nk$, with $n \in \{-(r - 1), -(r - 2), \dots, -1, 0, 1, \dots, s - 2, s - 1\}$, and each such $p + nk$ is prime. \square

In other words, every 2-chromatic graph is a prime distance graph. However, we note that not every 3-chromatic graph is a prime distance graph since $K_{3,3,3}$ is not a prime-distance graph. In fact, $K_{3,3,3}$ is not 2-odd.

Goldbach's Conjecture. [1, 15] *Every even number greater than 2 is the sum of two primes.*

Theorem 2. *If Goldbach's Conjecture is true, then every cycle is a prime distance graph.*

Proof. If $n = 3$, then C_n can be prime distance labeled with labels 0, 3, and 5. If $n = 4$ then C_n can be prime distance labeled with labels 0, 3, 8, and 11. If $n = 5$, then C_n can be prime distance labeled with labels 0, 3, 6, 9, and 11. Suppose $n \geq 6$, and write $2n - 4$ as the sum of two primes, say $2n - 4 = p_1 + p_2$. Then C_n can be prime distance labeled with labels 0, 2, $\dots, 2n - 4$, and p_1 in cyclic order. Since $2n - 4$ is even and at least 6, then p_1 must be odd, so the vertex labels are distinct. \square

Vinogradov's Theorem. [13, 16] *Every sufficiently large odd number is the sum of 3 primes.*

Theorem 3. *Every cycle is a prime distance graph.*

Proof. If $n = 3$, then C_n can be prime distance labeled with labels 0, 3, and 5. Suppose $n \geq 4$, and let p be a prime number large enough such that by Vinogradov's Theorem, $p + 2n - 8$ can be written as the sum of 3 primes, say $p + 2n - 8 = p_1 + p_2 + p_3$. Also assume $p > 4n$ and $p_1 \geq p_2 \geq p_3$, so in particular $p_1 > 2n - 8$. Then C_n can be prime distance labeled with labels $0, 2, \dots, 2n - 8, p + 2n - 8, p_1 + p_2$, and p_1 in cyclic order. Since $p_1 > 2n - 8$, the vertex labels are distinct. \square

Ramaré's Theorem. [14] *Every even number is the sum of at most 6 primes.*

Theorem 4. *Every cycle is a prime distance graph.*

Proof. If $3 \leq n \leq 7$ then C_n can be prime distance labeled using any of the techniques above. Suppose $n \geq 8$, let p be a prime number larger than $10n$, and write $2n - 5 + p$ as the sum of at most 6 primes, p_1 through p_i , where $2 \leq i \leq 6$. Again assume p_1 is the largest of these primes, so $p_1 > 2n - 5$. Then we have 5 cases:

Case 1. $2n - 5 + p = p_1 + p_2$. Then C_n can be prime distance labeled with labels $0, 2, 4, \dots, 2n - 8, 2n - 5, 2n - 5 + p$, and p_1 .

Case 2. $2n - 5 + p = p_1 + p_2 + p_3$. Then C_n can be prime distance labeled with labels $0, 2, 4, \dots, 2n - 10, 2n - 5, 2n - 5 + p, p_1 + p_2$, and p_1 .

Case 3. $2n - 5 + p = p_1 + p_2 + p_3 + p_4$. Then C_n can be prime distance labeled with labels $0, 2, 4, \dots, 2n - 12, 2n - 5, 2n - 5 + p, p_1 + p_2 + p_3, p_1 + p_2$, and p_1 .

Case 4. $2n - 5 + p = p_1 + p_2 + p_3 + p_4 + p_5$. Then C_n can be prime distance labeled with labels $0, 2, 4, \dots, 2n - 18, 2n - 15, 2n - 10, 2n - 5, 2n - 5 + p, p_1 + p_2 + p_3 + p_4, p_1 + p_2 + p_3, p_1 + p_2$, and p_1 .

Case 5. $2n - 5 + p = p_1 + p_2 + p_3 + p_4 + p_5 + p_6$. Then C_n can be prime distance labeled with labels $0, 2, 4, \dots, 2n - 20, 2n - 15, 2n - 10, 2n - 5, 2n - 5 + p, p_1 + p_2 + p_3 + p_4 + p_5, p_1 + p_2 + p_3 + p_4, p_1 + p_2 + p_3, p_1 + p_2$, and p_1 .

In each case, since $p_1 > 2n - 5$, the vertex labels are distinct. \square

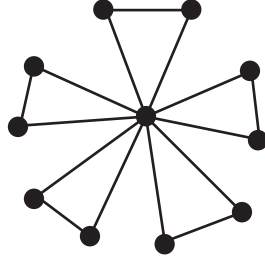


Figure 2: The Dutch windmill graph D_5 .

Recently, an announcement was made of a proof of the Weak Goldbach Conjecture [12]. The authors believe this theorem could also be used to prove that every cycle is a prime distance graph.

The **Dutch windmill graph** D_n or **friendship graph** is the star S_{2n} with central vertex v_0 and leaves v_1 through v_{2n} , with an edge between each consecutive pair of vertices v_{2k-1} and v_{2k} , $1 \leq k \leq n$. So D_n has n copies of C_3 joined at the common vertex v_0 [9, 8]. Figure 2 shows D_5 .

The Twin Prime Conjecture. There are infinitely many pairs of primes that differ by 2.

Theorem 5. *Every Dutch windmill graph is a prime distance graph if and only if the Twin Prime Conjecture is true.*

Proof. First assume that D_n has a prime distance labeling for any positive integer n , and consider one such prime distance labeling of D_n . Without loss of generality, assume that v_0 is labeled with 0. Note that there must be at most two more even labels on the remaining vertices, so at least $n - 2$ of the triangles in D_n have two odd labels. In each of these triangles, since both odd-labeled vertices are adjacent to v_0 , their labels must be prime. Since their difference is even and prime, it must be 2, so each triangle is labeled with a pair of twin primes. That is, if D_n is prime distance, there are at least $n - 2$ twin primes. Therefore, if all Dutch windmill graphs are prime distance graphs, then the Twin Prime conjecture is true.

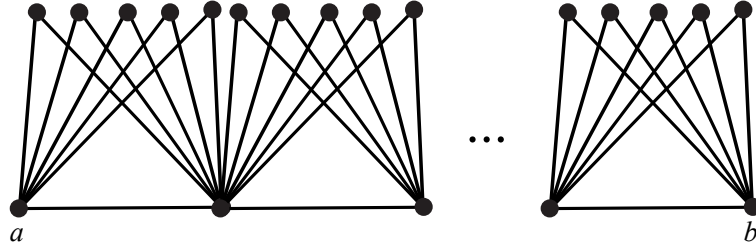


Figure 3: The stack of books S_k .

Conversely, if the Twin Prime Conjecture is true, then the labeling with 0 on v_0 and a pair of twin primes on each triangle is a prime distance labeling of D_n . \square

We construct the paper mill graphs as follows. First, the *(triangular) book graph* B_5 is the tripartite graph $K_{1,1,5}$ consisting of 5 triangles sharing a common edge, the *spine* of the book [18]. The *stack of books* S_k is a union of k copies of B_5 joined so that their spines form a path, as shown in Figure 3. Then let the *paper mill graph* $M_{n,k}$ be the graph constructed from D_n by replacing each edge uv not incident to the center vertex v_0 by a copy of S_k at vertices a and b .

dePolignac's Conjecture. [3, 2] *For any positive even integer $2k$, there exist infinitely many pairs of consecutive primes that differ by $2k$.*

Lemma 6. *In any prime distance labeling of S_k , the labels on the vertices a and b differ by exactly $2k$.*

Proof. By Corollary 10 below, in any prime distance labeling of B_5 , each 3-cycle must have exactly one edge labeled 2. But at most two of these edges can be incident to any single vertex, so the spine of B_5 must be labeled 2. Therefore all the edges in the path between a and b are labeled 2. \square

Theorem 7. *Every paper mill graph is a prime distance graph if and only if dePolignac's Conjecture is true.*

Proof. First assume that $M_{n,k}$ has a prime distance labeling for any positive integers n and k , and consider one such prime distance labeling of $M_{n,k}$. Without loss of generality, assume that the center vertex v_0 is labeled with 0. Then all the vertices adjacent to v_0 are labeled with primes. By Lemma 6, the ends of the paths on each S_k make up n pairs of primes such that each pair differs by $2k$. Therefore, if all paper mill graphs are prime distance graphs, dePolignac's Conjecture is true.

Conversely, assume that dePolignac's Conjecture is true; note that this also implies the Twin Prime Conjecture. We construct a prime distance labeling of $M_{n,k}$ for given positive integers n and k . Label v_0 with 0 and the i th path of spines by $p_i, p_i + 2, \dots, p_i + 2k$, where p_i and $p_i + 2k$ are prime. We choose each p_i so that p_i is sufficiently large that labels do not repeat. Then label the five additional vertices in the j th book with distinct integers so that the five pairs of edges are twin primes; do this for each book along S_i without repeating labels. (This is possible by assumption.) Repeat this for each S_i . \square

3. Edge Colorings

Given a prime distance (respectively, 2-odd) graph, we color red the edges labeled with 2 and blue the edges labeled with odd primes (odd integers). Conversely, given a graph G with edges colored red and blue (an **edge-colored** graph), a prime distance (respectively, 2-odd) labeling of G is **color-satisfying** if the label on every red edge is 2 and the label on every blue edge is prime (odd). We say that an edge-coloring of G is a **prime distance coloring** (respectively, **2-odd coloring**) if there exists a color-satisfying prime distance labeling (2-odd labeling) of G .

Suppose that G is edge-colored, and for two vertices u and v of G , the edge uv is red. If L is a color-satisfying prime distance (respectively, 2-odd) labeling of G , then $L(uv) = 2$, so we may assume without loss of generality that $L(u) = 0$ and $L(v) = 2$.

The following proposition shows that not every color-satisfying 2-odd labeling will lead to a prime-distance labeling. Throughout this paper, red and blue edges will be represented by thick and thin edges, respectively.

Proposition 8. *The graph G shown in Figure 4 is a prime distance graph but has a 2-odd coloring that is not a prime-distance coloring.*

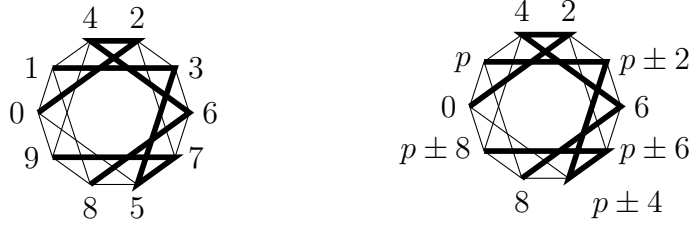


Figure 4: A color-satisfying 2-odd labeling and a failed attempt at a color-satisfying prime-distance labeling of the same graph

Proof. First note that by Theorem 27 below, G is a prime distance graph. Now the first labeling of G shown in Figure 4 is a color-satisfying 2-odd labeling. Note that G has two red paths, each with 5 vertices. If there were a prime-distance labeling L of G , without loss of generality, one of these paths would be labeled with 0, 2, 4, 6, and 8, and the other with p , $p \pm 2$, $p \pm 4$, $p \pm 6$, and $p \pm 8$, where p is a prime or the negative of a prime, as shown in the second labeling in Figure 4. However, note that p is adjacent to 0, 4, and 8, which constitute all of the equivalence classes modulo 3. Thus one of these three distances must be divisible by 3 and prime, and therefore exactly 3. So p must be one of the numbers -3 , 3, 1, 7, 5, or 11. But each of these values of p yields a non-prime edge: if $p = -3$, then $L((p+4)(0)) = 1$ and $L((p-4)(2)) = 9$; if $p = 3$ then $L((p)(4)) = 1$; if $p = 1$ then $L((p)(0)) = 1$; if $p = 7$ then $L((p)(8)) = 1$; if $p = 5$ then $L((p)(4)) = 1$; if $p = 11$ then $L((p-4)(8)) = 1$ and $L((p+4)(0)) = 15$. \square

Lemma 9. *Suppose G is an edge-colored graph with a color-satisfying 2-odd labeling L . Then every cycle in G has a positive even number of blue edges.*

Proof. Suppose $C = v_1v_2 \dots v_k$ is a cycle in G . If v_iv_{i+1} is red, then $L(v_i)$ and $L(v_{i+1})$ have the same parity, and if v_iv_{i+1} is blue, then $L(v_i)$ and $L(v_{i+1})$ have different parity. The parity of the labels of a cycle must change an even number of times, so C has an even number of blue edges. Now suppose by way of contradiction that C has no blue edges. Since v_1v_2 is red, we may assume without loss of generality that $L(v_1) = 0$ and $L(v_2) = 2$. Then since vertex labels are not repeated, $L(v_3) = 4$, $L(v_4) = 6$, \dots , $L(v_k) = 2k - 2$. Since $k \geq 3$, $L(v_1v_k) = 2k - 2$ is neither 2 nor odd. \square

Corollary 10. *In an edge-colored graph with a color-satisfying 2-odd labeling,*

every odd cycle has at least one red edge, and every 3-cycle has exactly one red edge.

Since every prime distance labeling is also a 2-odd labeling, Lemma 9 and Corollary 10 hold for prime distance labelings as well.

The following lemma will be useful also. Recall that the *symmetric difference* $H_1 \Delta H_2$ of two subgraphs H_1 and H_2 of a graph G is the graph with an edge e in $H_1 \Delta H_2$ if e is in H_1 or H_2 , but not both [18]. For convenience, we ignore isolated vertices in $H_1 \Delta H_2$. Note that if C_1 and C_2 are cycles in G , then $C_1 \Delta C_2$ is a union of edge-disjoint cycles in G .

Lemma 11. *If C_1 and C_2 are cycles in an edge-colored graph G that both have a positive even number of blue edges, then $C_1 \Delta C_2$ has a positive even number of blue edges, as well.*

Proof. If C_1 and C_2 have an odd number of blue edges in common, then they each have an odd number of blue edges that are not shared; thus, the number of blue edges in $C_1 \Delta C_2$ is even. If they have an even number of blue edges in common, then they also both have an even number that are not shared, and $C_1 \Delta C_2$ again has an even number of blue edges. \square

Note that by the result of Eggleton, Erdős, and Skilton above, $\chi(G) \leq 4$ if G is a prime distance graph. It follows that K_n is not a prime distance graph if $n \geq 5$. We prove that K_n is also not a 2-odd graph for $n \geq 5$.

Proposition 12. *The graph K_5 is not 2-odd.*

Proof. We label the vertices of K_5 as v_1, v_2, v_3, v_4 , and v_5 in cyclic order. By way of contradiction, assume that K_5 has a 2-odd coloring. By Lemma 9, the outside cycle of K_5 must have at least one red edge; without loss of generality, we may assume that v_1v_2 is red. Then by Corollary 10, v_1v_5 and v_2v_5 must be blue. Similarly, v_1v_4 and v_2v_4 must be blue, as must v_1v_3 and v_2v_3 . With both v_1v_3 and v_1v_4 blue, we must have v_3v_4 red again by Corollary 10. But now, because of 3-cycles on v_2, v_3, v_5 and v_3, v_4, v_5 , we must have v_3v_5 and v_4v_5 red, which creates a red 3-cycle on v_3, v_4 , and v_5 . \square

Since every subgraph of a 2-odd graph is also 2-odd, it follows that K_n is not 2-odd for $n \geq 5$.

4. A Characterization of 2-odd Graphs

The following theorem characterizes 2-odd graphs in terms of edge colorings. In an edge-colored graph, the *red-degree* (respectively, *blue-degree*) of a vertex v is the number of red edges (blue edges) incident with v .

Theorem 13. *A graph G is 2-odd if and only if it admits an edge-coloring satisfying the following two conditions:*

1. *No vertex of G has red-degree greater than 2.*
2. *Every cycle in G contains a positive even number of blue edges.*

Proof. If G is 2-odd, then it admits a labeling with distinct integers such that the difference between labels on adjacent vertices is either odd or exactly 2. This induces a coloring on the edges of G : when the difference between such labels is exactly 2, color the corresponding edge red; when the difference is odd, color the edge blue. No vertex can be incident with three or more red edges since such a vertex would have a label exactly 2 different from three distinct integers. Also, since the labeling is color-satisfying by construction, Lemma 9 implies that Condition 2 is satisfied.

Conversely, suppose that G admits an edge-coloring satisfying Conditions 1 and 2. Consider the subgraph H of G consisting solely of red edges. By Conditions 1 and 2, H is a union of paths. Consider the minor G/H obtained by contracting all of the red edges. We see that G/H is bipartite since G/H has no odd cycles by Condition 2. Label the partite sets of G/H A and B . Partition the vertices of G into sets A' and B' depending on whether they contract to a vertex in A or B in G/H . Note that since every red path in G contracts to a vertex in G/H , each red path in G is completely contained in either A' or B' . Label the vertices of each of these paths with consecutive odd integers if the path lies in A' and with consecutive even integers if the path lies in B' , using only previously unused integers in each case. Any remaining vertices of A' again can be labeled with any unused odd integers and remaining vertices of B' can be labeled with any unused even integers. This labeling is a 2-odd labeling of G since every edge between vertices of the same parity is labeled with 2. \square

5. Circulant Graphs

For a positive integer $n \geq 3$ and subset $S \subseteq \{1, 2, \dots, n\}$, the ***circulant graph*** $\text{Circ}(n, S)$ is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and an edge between vertices v_i and v_j if and only if $|i - j| \bmod n \in S$ [11]. Equivalently, $\text{Circ}(n, S)$ is the Cayley graph of the group \mathbb{Z}_n with generating set S . In this section we focus on the circulant graphs $\text{Circ}(n, \{1, k\})$ for $1 \leq k \leq n - 1$, which, for simplicity, we write as $\text{Circ}(n, k)$. Since $\text{Circ}(n, k) \cong \text{Circ}(n, n - k)$, we choose $k \leq n/2$.

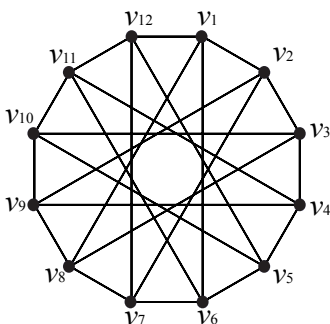


Figure 5: The graph $\text{Circ}(12, 5)$.

In these graphs we call the edges for which $|i - j| \bmod n = 1$ the ***outside edges*** and the edges for which $|i - j| \bmod n = k$ the ***inside edges***, since our drawings of $\text{Circ}(n, k)$ will have the outside edges around the outside of a circle, with inside edges as chords. Each vertex in $\text{Circ}(n, k)$ is incident to exactly 2 outside edges and 2 inside edges as long as $k > 1$. If $k = 1$ we say that $\text{Circ}(n, k)$ has no inside edges. In addition, we define a ***generating cycle*** of $\text{Circ}(n, k)$ to be either a $(k + 1)$ -cycle with exactly one inside edge, or the outside n -cycle. We will refer to the generating $(k + 1)$ -cycles as g_1, g_2, \dots, g_n , where g_i is the generating cycle with inside edge $v_i v_{i+k}$.

5.1. 2-odd Circulant Graphs

Lemma 14. *Every cycle in $\text{Circ}(n, k)$ is a generating cycle or a symmetric difference of generating cycles.*

Proof. Let C be a cycle in $\text{Circ}(n, k)$. The proof is by induction on the number of inside edges of C . If there are none, then C is the outside cycle and the lemma is satisfied. Now assume that C has c inside edges and

that any cycle with fewer than c inside edges satisfies the lemma. If C is a generating cycle, we are done. Otherwise, let D be a generating cycle whose inside edge also belongs to C . Then $C\triangle D$ is an edge-disjoint union of cycles C_1, C_2, \dots, C_k , each of which has fewer than c inside edges and therefore satisfies the lemma. Since $C_1\triangle C_2\triangle \dots \triangle C_k\triangle D = (C\triangle D)\triangle D = C$, we see that C also satisfies the lemma. \square

Theorem 15. *A circulant graph $\text{Circ}(n, k)$ is 2-odd if and only if it admits an edge-coloring satisfying the following two conditions:*

1. *No vertex of $\text{Circ}(n, k)$ has red-degree greater than 2.*
2. *Every generating cycle of $\text{Circ}(n, k)$ contains a positive even number of blue edges.*

Proof. Apply Lemmas 11 and 14 and Theorem 13. \square

Theorem 15 gives us a way to quickly verify whether a given edge-coloring of $\text{Circ}(n, k)$ shows that $\text{Circ}(n, k)$ is 2-odd.

The following theorem gives a characterization of 2-odd circulant graphs.

Theorem 16. *The circulant graph $\text{Circ}(n, k)$ is 2-odd if and only if $(n, k) \neq (5, 2)$.*

Proof. Case 1. $k = 1$. In this case $\text{Circ}(n, k) = C_n$, which by Theorem 3 is prime distance, hence 2-odd, for all n .

Case 2. n is even, k is even, and $1 < k < n/2$. We construct a 2-odd coloring of $\text{Circ}(n, k)$. Color red the outside edges $v_n v_1, v_k v_{k+1}, v_{2k} v_{2k+1}, \dots, v_{mk} v_{mk+1}$, where m is the greatest odd integer such that $mk < n$. That is, color red every k th outside edge, stopping after the maximum even number of outside edges are red without passing the first red edge. Also color red the inside edge of any generating cycle that contains both $v_n v_1$ and $v_{mk} v_{mk+1}$ (the first and last outside edges that are red) or none of the red outside edges. Color the remaining edges blue.

We show that this coloring satisfies both conditions of Theorem 15. We first check Condition 2. Note that our construction guarantees an even number of red edges on the outside cycle, which is an even cycle since n is even.

The remaining generating cycles are all odd, so to satisfy Condition 2 they must have an odd number of red edges. Since none of these cycles contain more than 2 outside red edges, by construction they have an inside red edge if and only if they have an even number of outside red edges, so Condition 2 is satisfied.

Finally, we check Condition 1. Since every vertex in $\text{Circ}(n, k)$ is incident to 2 outside edges and 2 inside edges, and there are no 2 consecutive red outside edges in this coloring, any vertex with red-degree 3 or greater must be incident to exactly 1 outside red edge and 2 inside red edges. However, the only red inside edges are those that belong to generating cycles containing both $v_n v_1$ and $v_{mk} v_{mk+1}$ or containing no outside red edges. The latter case cannot create a red-degree-3 vertex. For the former case, we observe that the only potential red-degree-three vertices are v_n , v_1 , v_{mk} , and v_{mk+1} . Of these, v_n and v_{mk+1} do not belong to generating cycles that contain both $v_n v_1$ and $v_{mk} v_{mk+1}$. Since neither $v_1 v_{k+1}$ nor $v_{(m-1)k} v_{mk}$ (the other inside edges on v_1 and v_{mk}) belongs to such a generating cycle either, none of the four candidates can have red-degree three.

Case 3. n is even, k is even, and $k = n/2$. Coloring exactly the inside edges red gives each generating cycle (except the outside cycle) one red edge and $k = n/2$ blue edges, and the outside cycle n blue edges. Thus, this edge-coloring satisfies Theorem 15.

Case 4. n is even and k is odd. Since every generating cycle has length $k + 1$ or length n , every cycle of G is even by Lemma 14, so coloring every edge of G blue yields a 2-odd coloring of G .

Case 5. n is odd and k is odd. Color the outside edge $v_1 v_2$ red, and color the k inside edges $v_{i-k} v_i$ red for $2 \leq i \leq k + 1$. This gives one red edge on the outside cycle and one red edge on each of the k generating cycles that include it. Note that the outside cycle is odd and the other generating cycles are even. This coloring gives 1 red edge on the outside cycle and either 0 or 2 red edges on each other generating cycle, satisfying the conditions of the theorem.

Case 6. n is odd, k is even, and $n > 5$. Apply the technique used in the proof of Case 2, but take m as the greatest *even* integer such that $mk < n$. The argument then proceeds as above. Note that the argument fails for $K_5 = \text{Circ}(5, 2)$ since the procedure would create a red 3-cycle.

Case 7. $(n, k) = (5, 2)$. In this case $\text{Circ}(n, k) = K_5$, which by Proposition 3 is not a 2-odd graph. □

We will use these 2-odd colorings to aid us in determining which circulant graphs are prime-distance graphs.

5.2. Prime Distance Circulant Graphs

The following theorem summarizes the results of this section. Recall that without loss of generality, we choose $k \leq n/2$.

Theorem 17. *The circulant graph $\text{Circ}(n, k)$ is not a prime distance graph if*

1. n is odd and $k = 2$
2. n is odd and $k = (n - 1)/2$.

The circulant graph $\text{Circ}(n, k)$ is a prime distance graph if

1. $n > 7$ and $k = 3$
2. n is even, $k = n/2$, and k is even
3. n is even and k is odd.

For the remaining values of n and k we make the following conjecture, which we have verified for $n \leq 14$.

Conjecture 18. *The circulant graph $\text{Circ}(n, k)$ is a prime distance graph if and only if none of the following hold:*

1. n is odd and $k = 2$
2. n is odd and $k = (n - 1)/2$
3. $(n, k) = (6, 2)$.

Lemma 19. *The edge-colored graph CF shown in Figure 6 has exactly one color-satisfying prime distance labeling up to isomorphism, labeling the vertices a , b , c , and d with 0, 2, 4, and 7, respectively.*

Proof. We construct a color-satisfying prime distance labeling L of CF . Since the edge ab is red, we may assume $L(a) = 0$ and $L(b) = 2$. Since labels on vertices are unique, this forces $L(c) = 4$.

Suppose $L(d) = x$. Then $L(A) = x$, $L(B) = |x - 2|$, and $L(C) = |x - 4|$ are all prime, so they must be the numbers 3, 5, and 7. Since there exists an automorphism swapping A and C , we may choose $L(A) = 7$, $L(B) = 5$, and $L(C) = 3$, so $L(d) = 7$. This is a valid color-satisfying prime distance labeling of CF . \square

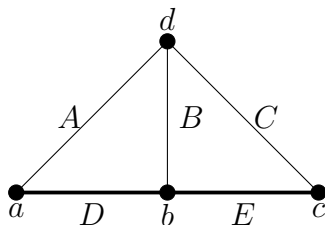


Figure 6: The edge-colored graph CF

We now define the edge-colored graph CF_k for all $k \geq 1$ (CF stands for colored fan). CF_k has vertices $\{a_1, a_2, \dots, a_{k+3}, b_1, b_2, \dots, b_k\}$ and is comprised of three paths and two extra edges: the red-edge path a_1, a_2, \dots, a_{k+3} , the red-edge path b_1, b_2, \dots, b_k , the blue-edge path $a_2, b_1, a_3, b_2, \dots, a_{k+1}, b_k, a_{k+2}$, and the two blue edges a_1b_1 and $a_{k+3}b_k$. The graphs CF_1 , CF_2 , and CF_k are shown in Figure 7.

Lemma 20. *CF_k has no color-satisfying prime distance labeling for all $k \geq 1$.*

Proof. By way of contradiction, suppose that CF_k has a color-satisfying prime distance labeling L . The induced subgraph of CF_k on the vertices a_1, a_2, a_3 , and b_1 is CF , so by Lemma 19, $L(a_1) = 0$, $L(a_2) = 2$, $L(a_3) = 4$, and $L(b_1) = 7$. In the case $k = 1$, this forces a_4 to have label 6, which makes a_4b_1 non-prime. Now assume $k > 1$.

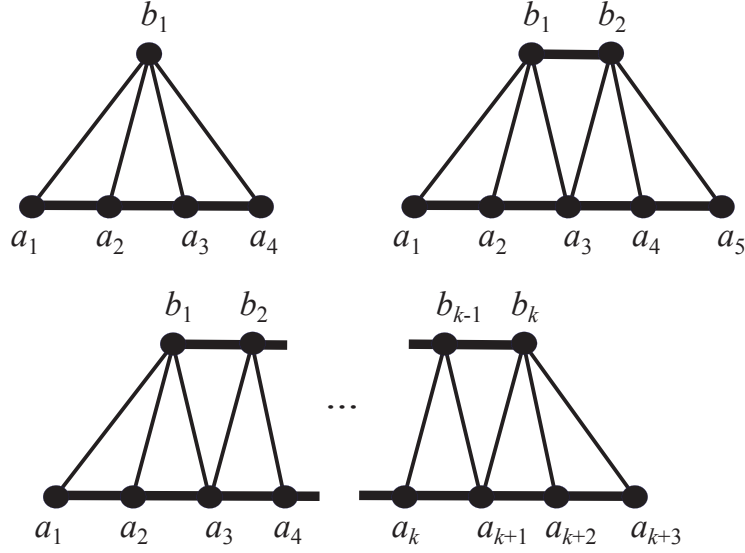


Figure 7: The edge-colored graphs CF_1 , CF_2 , and CF_k .

Since edge b_1b_2 is red, $L(b_2)$ is either 5 or 9. But if $L(b_2) = 5$ then $L(a_3b_2) = 1$, which is not prime, so $L(b_2) = 9$. Since the edges a_3a_4 , a_4a_5 , \dots , $a_{k+2}a_{k+3}$ and the edges b_2b_3 , b_3b_4 , \dots , $b_{k-1}b_k$ are all red, this forces the labeling on the remaining vertices $L(a_4) = 6$, $L(a_5) = 8$, \dots , $L(a_{k+3}) = 2k + 4$ and $L(b_3) = 11$, $L(b_4) = 13$, \dots , $L(b_k) = 2k + 5$. But this means $L(a_{k+3}b_k) = 1$, which is not prime.

□

Theorem 21. $\text{Circ}(2n + 1, 2)$ is not a prime distance graph for all $n \geq 2$.

Proof. Suppose by way of contradiction that L is a prime distance labeling of $\text{Circ}(2n + 1, 2)$. Since the outside edges of $\text{Circ}(2n + 1, 2)$ form an odd cycle and the inside edges form an odd cycle, by Lemma 10, $\text{Circ}(2n + 1, 2)$ must have at least one red outside edge and at least one red inside edge.

Suppose without loss of generality that the edge v_1v_2 is red, and $L(v_1) = 0$ and $L(v_2) = 2$. Note that v_2v_3 must be blue by Lemma 9 since vertices v_1 , v_2 , and v_3 form a 3-cycle. Suppose v_iv_{i+1} is the next red outside edge, i.e. v_iv_{i+1} is red and v_jv_{j+1} is blue for all $2 \leq j \leq i - 1$. Consider the induced subgraph G of $\text{Circ}(2n + 1, 2)$ on the vertices v_1, \dots, v_{i+1} . If i is even (and thus $i \geq 4$), we will show that the coloring on the edges of G induced by

the prime distance labeling L of $\text{Circ}(2n+1, 2)$ yields the edge-colored graph $CF_{i/2-1}$. Since $CF_{i/2-1}$ has no color-satisfying prime distance labeling, this will prove that i must be odd.

Since v_1v_2 is red, v_1v_3 must be blue by Corollary 10. Since v_iv_{i+1} is red, $v_{i-1}v_{i+1}$ must be blue by Corollary 10. For all $2 \leq j \leq i-2$, since v_jv_{j+1} and $v_{j+1}v_{j+2}$ are blue, v_jv_{j+2} must be red by Corollary 10. This is exactly the edge-coloring of the graph $CF_{i/2-1}$, with $v_1 = a_1, v_2 = a_2, v_3 = b_1, v_4 = a_3, v_5 = b_2, v_6 = a_4, \dots, v_{i-1} = b_k, v_i = a_{k+2}, v_{i+1} = a_{k+3}$.

Thus i must be odd. Note that vertices with a red edge between them have labels with the same parity, and vertices with a blue edge between them have labels with different parity. Since the number of blue edges between v_2 and v_i is $i-2$, which is odd, $L(v_i)$ is odd, so $L(v_{i+1})$ is also odd. Continuing in this way, if v_jv_{j+1} is the next red edge, $L(v_j)$ is even and $L(v_{j+1})$ is even. Thus the labels on the vertices of the outside red edges alternate parity in pairs. This implies that there are an even number of outside red edges, contradicting Lemma 10. □

Since we know that $\text{Circ}(2n+1, 2)$ is a 2-odd graph if $n > 2$, we also obtain the following corollary.

Corollary 22. *Not every 2-odd graph is a prime distance graph.*

A **bowtie path** in $\text{Circ}(2n+1, n)$ is a path of the form

$$v_a v_{a+n+1} v_{a+n+2} v_{a+n+3} \cdots v_{a+n+r} v_{a+r}$$

for some positive integers a and r . Figure 8 shows an example of a bowtie path.

Lemma 23. *If in a 2-odd coloring of $\text{Circ}(2n+1, n)$, the edges $v_a v_{a+1}$ and $v_{a+1} v_{a+2}$ are red, then so is the edge $v_{a+n+1} v_{a+n+2}$.*

Proof. Without loss of generality, let $a = 1$. Since v_1v_2 is red, we must have both v_1v_{n+2} and v_2v_{n+2} blue since v_1, v_2 , and v_{n+2} form a triangle. Similarly, since v_2v_3 is red, v_2v_{n+3} and v_3v_{n+3} are blue. But then we have v_2v_{n+2} and v_2v_{n+3} both blue, so $v_{n+2}v_{n+3}$ must be red since it is the third edge in the triangle with vertices v_2, v_3 , and v_{n+3} . □

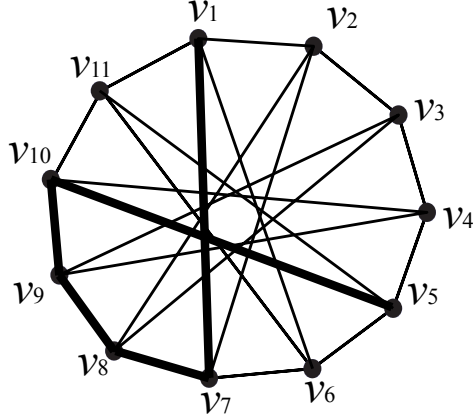


Figure 8: A bowtie path in $\text{Circ}(2n+1, n)$, with $n = 5$, $a = 1$, and $r = 4$.

Lemma 24. *If $\text{Circ}(2n+1, n)$ is edge-colored so that a bowtie path in $\text{Circ}(2n+1, n)$ has only red edges, then it has no color-satisfying prime-distance labeling.*

Proof. Suppose by way of contradiction that $\text{Circ}(2n+1, n)$ has such a prime-distance labeling. Without loss of generality, let $a = 1$, so the all-red bowtie path in $\text{Circ}(2n+1, n)$ is on the vertices $v_1, v_{n+2}, v_{n+3}, \dots, v_{n+r+1}, v_{r+1}$. Again without loss of generality, we may assume that these vertices are labeled $0, 2, 4, 6, \dots$, and $2(r+1)$. By Lemma 23, the edges $v_{n+3+n}v_{n+4+n} = v_2v_3, \dots, v_{n+r+n}v_{n+r+1+n} = v_{r-1}v_r$ are also red. Note that if $r = 2$ then there are no edges of this form. In either case, the edge v_1v_2 must be blue since it belongs to the triangle with vertices v_1, v_2, v_{n+2} which already has the red edge v_1v_{n+2} .

Consider the possible labels on the vertex v_2 . This vertex is adjacent to vertices with labels $0, 2$, and 4 , which represent all equivalence classes modulo three. Thus one of the distances between v_2 and these three neighbors must be a multiple of 3 and prime, so it must be exactly three. Thus, v_2 must be labeled with either -3 or 7 . Now we may proceed along the new red path given by Lemma 23.

The next label along this path after a -3 could be -5 since it is adjacent to a vertex labeled 4 , and the next label after a 7 could be 5 for the same reason. Thus the red path beginning at v_2 must have labels $-3, -1, 1, \dots, 2r-7$ or $7, 9, 11, \dots, 2r+3$. But the terminal vertex in this path, which

is labeled either $2r - 7$ or $2r + 3$, is adjacent to v_{r+1} , which is labeled with $2(r+1)$. The distance between these two vertices is thus either 9 or 1, neither of which is prime. \square

Define the **red path** RP_m and the **blue path** BP_m as the edge-colored subgraphs of $\text{Circ}(2n + 1, n)$ shown in Figure 9.

Theorem 25. $\text{Circ}(2n + 1, n)$ is not a prime distance graph for all $n \geq 2$.

Proof. Suppose by way of contradiction that $\text{Circ}(2n + 1, n)$ is a prime distance graph, and consider a prime distance labeling of $\text{Circ}(2n + 1, n)$, with corresponding edge coloring. Since $2n + 1$ is odd, the outside cycle must have at least one red edge.

Step 1. Every maximal red outside path in $\text{Circ}(2n + 1, n)$ with vertices v_i, \dots, v_{i+m} induces the edge-colored subgraph RP_m shown in Figure 9 as follows:

Since this red path is maximal, the edges $v_{i-1}v_i$ and $v_{m+i}v_{m+i+1}$ are blue. By Corollary 10, this induces blue edges v_iv_{n+i+1} , $v_{i+1}v_{n+i+1}$, $v_{i+1}v_{n+i+2}$, $v_{i+2}v_{n+i+2}$, \dots , $v_{m+i-1}v_{m+i+n}$, $v_{m+i}v_{m+i+n}$ and red edges $v_{n+i+1}v_{n+i+2}$, $v_{n+i+2}v_{n+i+3}$, \dots , $v_{m+i+n-1}v_{m+i+n}$, as well as $v_{i-1}v_{n+i}$. By Lemma 24, we cannot have both edges $v_{m+i}v_{m+i+n+1}$ and v_iv_{n+i} red, so we may assume without loss of generality, that v_iv_{n+i} is blue. Since v_iv_{n+i+1} is also blue, we must have $v_{n+i}v_{n+i+1}$ red. If $v_{m+i}v_{m+i+n+1}$ were blue as well, then, because $v_{m+i}v_{m+i+n}$ is blue, we would additionally have $v_{m+i+n}v_{m+i+n+1}$ and $v_{m+i+n+1}v_{m+i+1}$ red, contradicting Lemma 24. Thus $v_{m+i}v_{m+i+n+1}$ must be red and $v_{m+i+n}v_{m+i+n+1}$ must be blue.

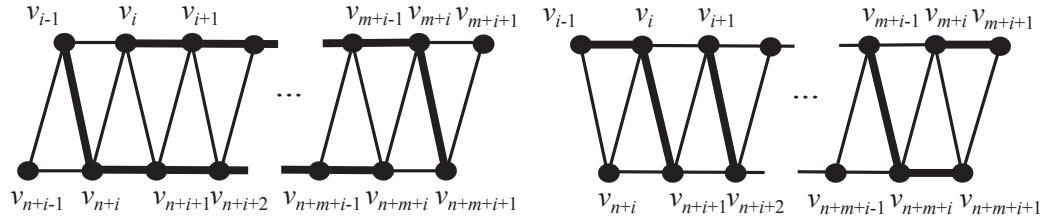


Figure 9: The edge-colored graphs RP_m (left) and BP_m (right).

Step 2. Every maximal blue outside path in $\text{Circ}(2n + 1, n)$ with vertices v_i, \dots, v_{m+i} induces the edge-colored subgraph BP_m shown in Figure 9.

Since this blue path is maximal, the edges $v_{i-1}v_i$ and $v_{m+i}v_{m+i+1}$ are red. By Step 1, either the edge v_iv_{n+i+1} or the edges $v_{n+i}v_{n+i+1}$ and $v_{n+i+1}v_{i+1}$ are red. Assume v_iv_{n+i+1} is red (the other case is a reflection of this one). Then by Corollary 10, $v_{i+1}v_{n+i+1}$ is blue. If $v_{i+1}v_{i+2}$ is red, then $v_{i+1}v_{n+i+2}$ is blue, $v_{n+i+1}v_{n+i+2}$ is red, and we have BP_1 . Otherwise we continue as before, until reaching the first red edge $v_{m+i}v_{m+i+1}$.

Step 3. The subgraphs RP_m and BP_m cannot fit together to form a valid red-blue edge coloring of $\text{Circ}(2n + 1, n)$: to complete the outside cycle (after gluing together whatever copies of BP_m and RP_m appear), the top left corner of RP_m must be adjacent to the bottom right corner of BP_m . However, this results in a red bowtie path:

$$\underbrace{v_{m+i-1}v_{n+m+i}v_{n+m+i+1} \dots v_{n+m+i+r}}_{\text{from end of } BP_m} \quad \underbrace{v_{n+m+i+r+1}v_{m+i+r}}_{\text{from beginning of } RP_m} \quad .$$

Therefore $\text{Circ}(2n + 1, n)$ is not a prime-distance graph. \square

Proposition 26. *If n is even and k is odd, then $\text{Circ}(n, k)$ is a prime distance graph.*

Proof. Since $\text{Circ}(n, k)$ is bipartite if n is even and k is odd, the result follows from Theorem 1. \square

Theorem 27. *If $n > 7$, then $\text{Circ}(n, 3)$ is a prime distance graph.*

Proof. By the previous Proposition, it suffices to consider $\text{Circ}(2n + 1, 3)$. The labeling depends on the congruence class of n modulo 3.

Case 1. $n \equiv 0 \pmod{3}$. In this case we label the vertices $v_1, v_2, v_3, v_4, v_5, v_6, \dots, v_{3i+1}, v_{3i+2}, v_{3i+3}, \dots, v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, v_{n+5}, \dots, v_{3j}, v_{3j+1}, v_{3j+2}, \dots, v_{2n-3}, v_{2n-2}, v_{2n-1}, v_{2n}$, and v_{2n+1} with the labels $4, 2, -1, 6, 9, 12, \dots, 11i - 5, 11i - 2, 11i + 1, \dots, \frac{11}{3}n - 21, \frac{11}{3}n - 16, \frac{11}{3}n - 13, \frac{11}{3}n - 2, \frac{11}{3}n + 1, \frac{11}{3}n + 4, \frac{11}{3}n - 9, \frac{11}{3}n - 12, \frac{11}{3}n - 15, \dots, \frac{22}{3}n - 11j + 2, \frac{22}{3}n - 11j - 1, \frac{22}{3}n - 11j - 4, \dots, 13, 10, 7, 0,$ and -3 , respectively. Note that the edges around the outside cycle appear in the

pattern $3, 3, 5, 3, 3, 5, \dots$, except for the edges between vertices v_{2n-1} through v_4 and vertices v_{n-1} through v_{n+3} . Hence most inner edges are labeled 11. The exceptions are easily verified to be prime also.

Case 2. $n \equiv 1 \pmod{3}$. In this case we keep the majority of the labels from Case 1, but change the labels of the vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, v_{n+5}, v_{n+6}, \dots, v_{3j}, v_{3j+1}$, and v_{3j+2} to $\frac{11(n-1)}{3} - 16, \frac{11(n-1)}{3} - 13, \frac{11(n-1)}{3} - 10, \frac{11(n-1)}{3} + 7, \frac{11(n-1)}{3} + 10, \frac{11(n-1)}{3} - 7, \frac{11(n-1)}{3} - 4, \frac{11(n-1)}{3} - 9, \frac{11(n-1)}{3} - 12, \frac{11(n-1)}{3} - 15, \dots, \frac{22(n-1)}{3} - 11j + 10, \frac{22(n-1)}{3} - 11j + 7$, and $\frac{22(n-1)}{3} - 11j + 2$, respectively. Again edges around the outside cycle appear in the pattern $3, 3, 5, 3, 3, 5, \dots$, and the exceptions are easily verified to be prime also.

Case 3. $n \equiv 2 \pmod{3}$. In this case again we keep the majority of the labels from Case 1, but change the labels of the vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, v_{n+5}, \dots, v_{3j}, v_{3j+1}$, and v_{3j+2} to $\frac{11(n-2)}{3} - 11, \frac{11(n-2)}{3} - 8, \frac{11(n-2)}{3} - 5, \frac{11(n-2)}{3} + 6, \frac{11(n-2)}{3} + 9, \frac{11(n-2)}{3} + 2, \frac{11(n-2)}{3} - 1, \frac{11(n-2)}{3} - 4, \frac{11(n-2)}{3} - 9, \dots, \frac{22(n-2)}{3} - 11j + 18, \frac{22(n-2)}{3} - 11j + 13$, and $\frac{22(n-2)}{3} - 11j + 10$, respectively. Again the proof goes through as in Case 1. \square

The proof of Theorem 27 illustrates the difficulty in characterizing prime distance circulant graphs. We believe this technique could be modified to prove $\text{Circ}(n, k)$ is a prime distance graph for a fixed k strictly between 2 and $(n-1)/2$, but a new technique is needed to prove this for general k .

Theorem 28. *If $n > 2$ and $n/2$ are even, then $\text{Circ}(n, n/2)$ is a prime distance graph.*

Proof. $\text{Circ}(4, 2)$ and $\text{Circ}(8, 4)$ are easily shown to be prime distance, so assume $n \geq 12$. Choose a prime p sufficiently large such that by Vinogradov's Theorem, $p - 3n/2 + 10$ can be expressed as a sum of three primes q_1, q_2, q_3 . That is, $p = 3n/2 + 10 + q_1 + q_2 + q_3$.

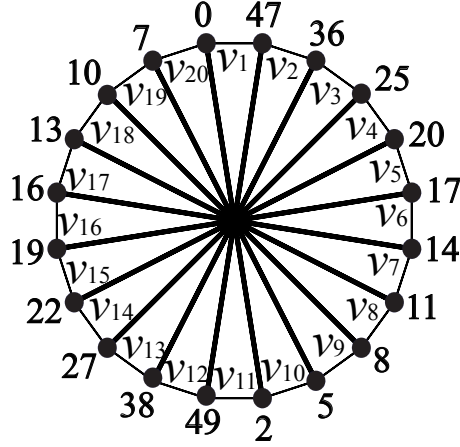


Figure 10: A prime distance labeling of $\text{Circ}(20, 10)$

In counterclockwise order, we label v_n with 0, and $v_{n-1}, v_{n-2}, v_{n-3}, \dots, v_{n/2+4}$ with 7, 10, 13, $\dots, 3n/2 - 8$, respectively, label $v_{n/2}$ with 2, and $v_{n/2-1}, v_{n/2-2}, v_{n/2-3}, \dots, v_4$ with 5, 8, 11, $\dots, 3n/2 - 10$, respectively.

Then we label v_1 with p , v_2 with $q_1 + q_2 + 3k - 10$, v_3 with $q_1 + 3k - 10$, $v_{n/2+1}$ with $p+2$, $v_{n/2+2}$ with $q_1 + q_2 + 3n/2 - 8$, and $v_{n/2+3}$ with $q_1 + 3n/2 - 8$. Figure 10 shows the case $n = 10$ and $p = 47$.

This gives a prime distance labeling of G . \square

6. Open Questions

In addition to Conjecture 18, we pose the following open questions.

1. Is there a family of graphs which are prime distance graphs if and only if Goldbach's Conjecture is true?
2. What circulant graphs $\text{Circ}(n, S)$, for more general sets S , are prime distance graphs?
3. By our definition of prime distance graphs, $L(uv)$ may still be prime if uv is not an edge of G . How do the results of this paper change if we define $L(uv)$ to be prime if and only if uv is an edge of G ?
4. What other families of graphs are prime distance graphs? More specifically, by Eggleton, Erdős, and Skilton's result, all prime distance graphs

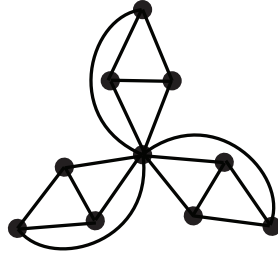


Figure 11: A planar graph which is not a prime distance graph.

have chromatic number at most 4, but not all planar graphs are prime distance graphs by the example shown in Figure 11. Can we classify which planar graphs are prime distance graphs?

7. Acknowledgements

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