Seeing Dots:
Explorations on the Visibility of Lattice Points

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Introduction

Consider a photographer who is commissioned to take publicity pictures of a high school marching band. He wishes to take the smallest number of photographs such that every face is visible and unobstructed in at least one of the photographs, so as not to upset any member of the PTA.

A marching band's traditional formation is in a grid pattern, with exactly five feet between each student, horizontally and vertically as viewed from above. We abstract the problem by representing each member of the band by an integer lattice point, and say that she is viewed in the photograph if there is no other student on the line segment between her and the photographer. Notice that this is a simplification, since some lines of sight will pass very close to another lattice point without passing through it. We represent the band as an $r \times s$ rectangle of integer lattice points with corners at $(1, 1)$, $(r, 1)$, $(1, s)$, and (r, s), denoted by $\Delta_{r,s}$, and the photographer as a small number of integer lattice points outside the band formation, one for each picture. An example of a particular picture is shown in Figure 1. Note that the photographer captures three of the four students in this shot.

Formally, two distinct integer lattice points P and Q are *mutually visible* if there are no other integer lattice points on the line segment joining P and Q. Alternatively, $P = (a_1, a_2)$ and $Q = (b_1, b_2)$ are mutually visible if and only if $\gcd(a_1 - b_1, a_2 - b_2) = 1$ [7]. We will use this alternative definition more frequently in our arguments.

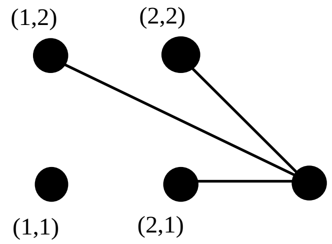

Figure 1: From the point $(3, 1)$, the photographer can see three of the four faces in this band.

Weak Lattice Point Visibility

As a warm-up question, we ask where the photographer might position himself to take a single photograph in an empty field, so that every face is clearly visible in the photograph. We say that the rectangle of lattice points $\Delta_{r,s}$ is *weakly visible* from the point P if no line segment between P and a point in $\Delta_{r,s}$ passes through another point in $\Delta_{r,s}$. It turns out that it takes only a single point to weakly view a rectangle of any size.

Theorem 1. *$\Delta_{r,s}$ is weakly visible from the point $P = (rs - s + r, s + 1)$.*

Proof. Consider the set of all lines $\mathcal{L}$ between pairs of points in $\Delta_{r,s}$. We claim that P is not on any line in $\mathcal{L}$. If this claim were true, the theorem would follow.

The line L connecting the points $(1, 1)$ and $(r, 2)$ is the line with the smallest positive slope in $\mathcal{L}$. The line H with equation $y = s$ is the horizontal line with largest y-intercept in $\mathcal{L}$. Therefore, no lines in $\mathcal{L}$ cross through the region below L and above H. But since L contains the point $(rs-s+r, s+2)$ and H contains the point $(rs - s + r, s)$, P is below L and above H, as illustrated in Figure 2. □

It is interesting to note that the situation only improves in higher dimensions. If A, B, and C are three points in n-dimensional space, and their projections onto the $x - y$ plane are not collinear, then A, B, and C are

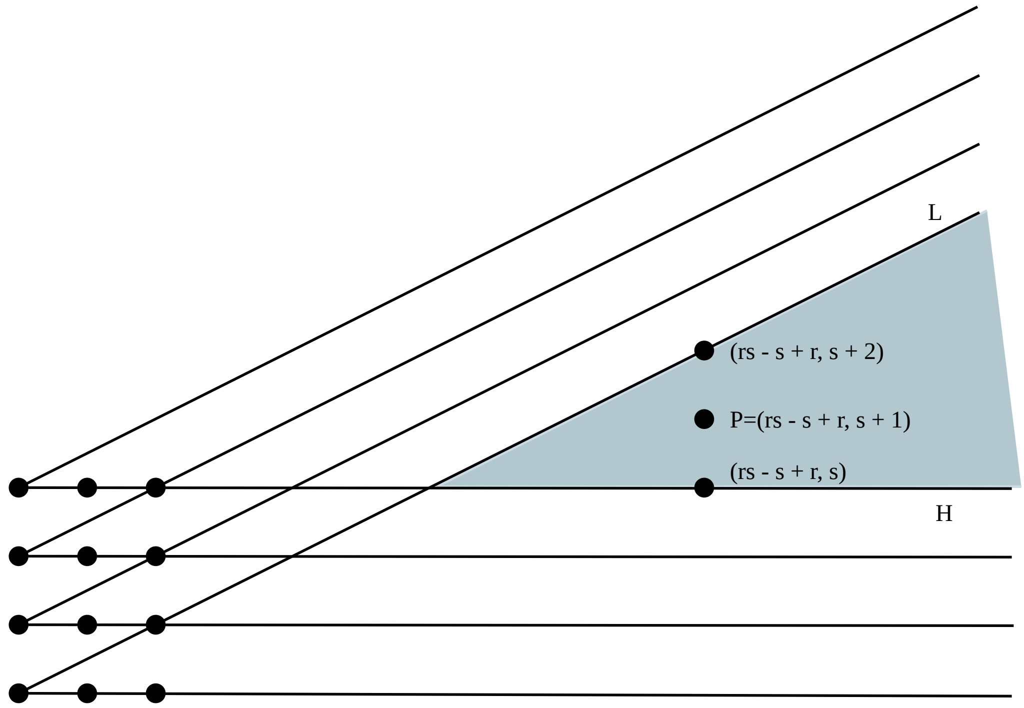

Figure 2: The point $(rs - s + r, s + 1)$ weakly views all of $\Delta_{r,s}$.

also not collinear. Therefore if Δ is an n-dimensional box of integer lattice points, with x and y dimensions r and s, respectively, then Δ is still weakly visible from the point $P = (rs - s + r, s + 1, 0, 0, \ldots, 0)$.

Also note that the point we have found might be quite far from the rectangle or box we wish to view, and so the faces in the photograph might be quite small.

Problem 2. *Find the point(s) which weakly view the rectangle $\Delta_{r,s}$ (or higher-dimensional box Δ) and are closest to it.*

External Lattice Point Visibility

Now that we have answered our question in an empty field, we turn to a more interesting question: What if the band is marching down a crowded street?

We can represent the crowd by adding in the rest of the integer lattice points in the plane. Now to view a member of the band, the photographer must have a line of sight that doesn't pass through any member of the band or of the crowd. Again, we ask how few photographs are needed to view every face in the band. Note that this question is not interesting if we are allowed to take our photographs from non-integer points, so we restrict ourselves to integer points for this reason.

Formally, we say that $\Delta_{r,s}$ is *externally visible* from the integer lattice points P_1, P_2, ..., P_k not in $\Delta_{r,s}$, if for every point Q in $\Delta_{r,s}$, Q and P_i are mutually visible for some i. We then define $e(r, s)$ to be the smallest number of points outside $\Delta_{r,s}$ from which $\Delta_{r,s}$ is externally visible. We are interested in the exact value of $e(r, s)$ for all positive integers r and s.

As a useful visualization technique, we consider a single photograph of an arbitrarily large band, with the photographer standing off of the lower-left-hand corner, at the point $(0, 0)$. Figure 3 shows the points that are visible from this position as black squares, and the points that are not visible as white squares. A photograph taken from any vantage point will see the same pattern of visible and non-visible points, starting at the chosen point instead of $(0, 0)$, and possibly reflected horizontally or vertically depending on the direction the camera is pointing. Thus, we can simulate multiple photographs by overlaying several copies of Figure 3, copied onto transparency paper.

Others have looked at similar questions about lattice point visibility. Abbott defined $f(n)$ to be the smallest number of points in $\Delta_{n,n}$ required to view the rest of $\Delta_{n,n}$, and proved that $\frac{\log n}{2 \log \log n} < f(n) < 4 \log n$ for large n [1]. Chen and Cheng defined $g(n)$ to be the smallest number of lattice points in the plane, inside or outside of $\Delta_{n,n}$, required to view the rest of $\Delta_{n,n}$, and proved that $g(n) \geq \frac{k \log n}{\log \log n}$ where k approaches $\frac{\pi^2}{6}$ [5].

We start with some general comments. By symmetry, $e(r, s) = e(s, r)$. Seeing $\Delta_{r,s}$ means seeing every subset of $\Delta_{r,s}$, therefore $e(r, s) \leq e(t, u)$ if $r \leq t$ and $s \leq u$. The point $(0, 0)$ can view any point of the form $(1, k)$, which means that $e(1, s) = 1$ for all positive integers s.

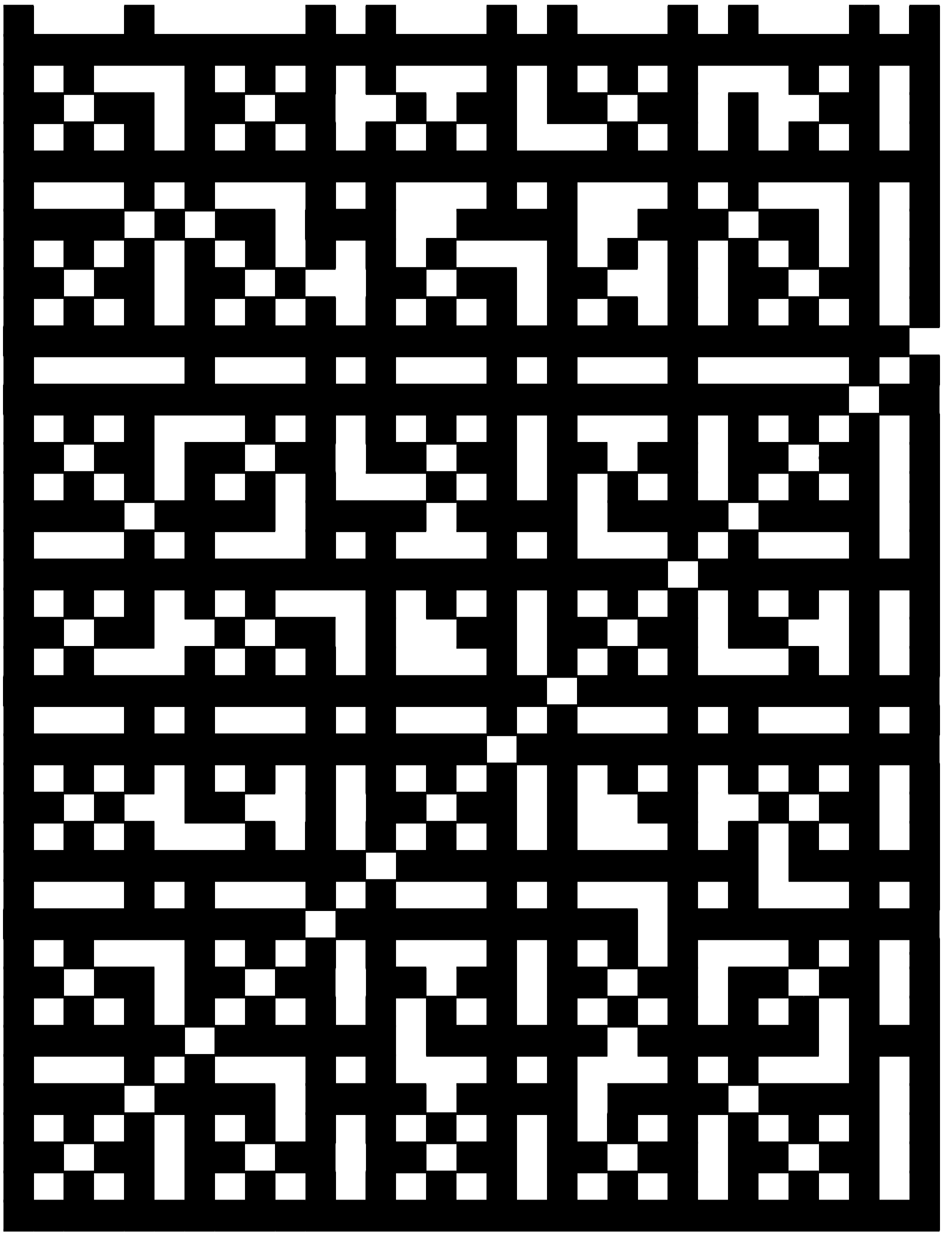

Figure 3: Lattice points visible from the point $(0, 0)$ appear as black squares, and nonvisible points appear as white squares.

However, $e(2,2) = 2$. To see this, let (a,b) be an integer lattice point not in $\Delta_{2,2}$. Each of the two integers a and b is either odd or even. Suppose that a is odd and b is even. Then $a-1$ and $b-2$ are both even, so (a,b) and $(1,2)$ are not mutually visible. In an analogous way, every point (a,b) cannot view at least one of the points in $\Delta_{2,2}$, so $e(2,2) > 1$. The points $(0,0)$ and $(0,1)$ externally view $\Delta_{2,2}$, so $e(2,2) = 2$.

We can generalize this idea to larger rectangles. Herzog and Stewart [6] defined a *pattern* P to be a subset of integer lattice points in $\mathbb{R}^2$. They defined a particular pattern P to be *realizable* if P can be translated by some vector in $\mathbb{R}^2$ such that every point in P is visible from the origin. In other words, P is realizable if it can be externally viewed from a single point. They defined a *complete square modulo* m to be a set $S = \{(x_k, y_k)\}$ of m^2 integer lattice points in $\mathbb{R}^2$, such that $\{(x_k \mod m, y_k \mod m)\} = \{(a,b) | 0 \leq a, b \leq m-1\}$. Given these definitions, they proved the following theorem.

Theorem 3 (Herzog and Stewart, 1971). *A given pattern P is realizable if and only if P fails to contain a complete square modulo p for every prime p.*

We can use this theorem to demonstrate that two points are not enough to externally view $\Delta_{6,6}$. Let $P = (a_1, a_2)$ and Q be any two integer lattice points outside of $\Delta_{6,6}$. We consider the set $S = \{(a_1 + 3j, a_2 + 3k) | j, k \in \mathbb{Z} \text{ not both } 0\}$. No point in S is visible from P. There are exactly four points in both S and $\Delta_{6,6}$, and they form a complete square modulo 2. By Theorem 3, these four points cannot all be externally viewed from Q. Therefore, $e(6,6) > 2$.

Not only that, a 6×6 rectangle is the smallest which requires three points. Consider the points $(0,0)$ and $(0,-1)$. If a and b are positive integers, $(0,0)$ is mutually visible with any point (a,b), where a and b are relatively prime, and $(0,-1)$ is mutually visible with any point (a,b), where a and $b-1$ are relatively prime. If a is an integer between 1 and 5, then for every positive integer b, either a and b are relatively prime, or a and $b-1$ are relatively prime. Therefore $e(5,s) = 2$ for all positive integers s. We might also choose to define $\Delta_{5,\infty}$ as the set of all lattice points with x-coordinate between 1 and 5, and positive y-coordinate. Then by the same reasoning, $e(5,\infty) = 2$.

We now know exactly which rectangles can be viewed from two external points: the rectangles with less than six points along one of their sides. Continuing our search for values of $e(r,s)$, our next question is, what is the largest rectangle viewable from three external points?

We try adding a third point to the two we already have, and we find that the points $(0,0)$, $(0,-1)$, and $(15,-2)$ externally view $\Delta_{14,s}$ for any positive integer s. We have already seen that $(0,0)$ and $(0,-1)$ view all points (a,b) with either $\gcd(a,b) = 1$ or $\gcd(a,b-1) = 1$. The points in $\Delta_{14,s}$ not satisfying either of these criteria have one of the following forms: $(6,6n+3)$, $(6,6n+4)$, $(10,10n+5)$, $(10,10n+6)$, $(12,6n+3)$, $(12,6n+4)$, $(14,14n+7)$, or $(14,14n+8)$, where n is a non-negative integer. These points are shown as squares in Figure 4. By brute force, we can check that every one of these points is visible from $(15,-2)$.

We display a picture of $\Delta_{14,15}$ in Figure 4. The circles in the figure represent those points that cannot be viewed from $(0,0)$. The first set of line segments illustrate those points that can be viewed from $(0,-1)$. The squares are the points that still remain. The second set of line segments illustrate that these points are visible from $(15,-2)$.

Using the same ideas we used to show that $e(2,2) > 1$ and $e(6,6) > 2$, we can prove a general lower bound for $e(r,s)$.

Theorem 4. *Let p_i be the i^{th} prime number, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on, and let $n = p_1 p_2 \dots p_i$. Then $e(n,n) > i$.*

Proof. Let $P_1 = (x_1,y_1)$, …, $P_i = (x_i,y_i)$ be i points not in $\Delta_{n,n}$. By the Chinese Remainder Theorem, there exist numbers a and $b \leq n$ such that $x_k \equiv a \mod p_k$, and $y_k \equiv b \mod p_k$, for each k between 1 and i. Therefore $x_k - a$ and $y_k - b$ are both divisible by p_k, and so (a,b) is not externally visible from any of the points P_1 through P_i.

□

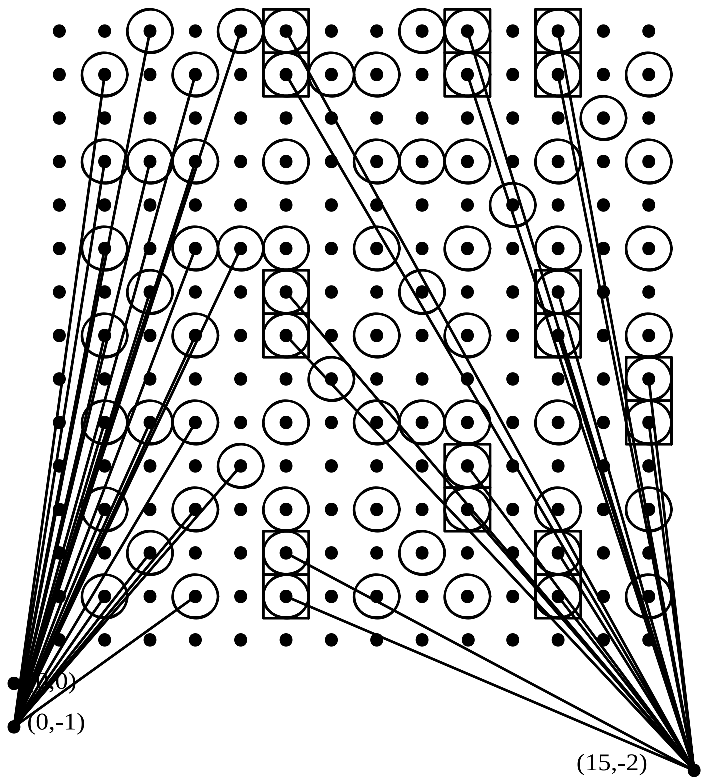

Figure 4: Viewing $\Delta_{14,15}$ from the points $(0,0)$, $(0,-1)$, and $(15,-2)$.

In particular, Theorem 4 tells us that $e(30,30) > 3$. Figure 5 shows the connection between Theorem 3 and Theorem 4. The circles in the figure represent those points that cannot be externally viewed from an arbitrary single point (this is the same pattern that appears in Figure 3). The light gray circles pick out a pattern that includes 25 complete squares modulo 2. Any second external point must miss one point from each of these complete squares, indicated by the dark grey circles. But notice that the dark grey circles form a complete square modulo 5, requiring at least two additional points to view the entire rectangle.

So $e(14,s) = 3$, and $e(30,30) > 3$. We remark that $(6,0)$, $(6,-1)$, and $(5,-16)$ externally view $\Delta_{23,19}$, so $e(23,19) = 3$, but these points do not

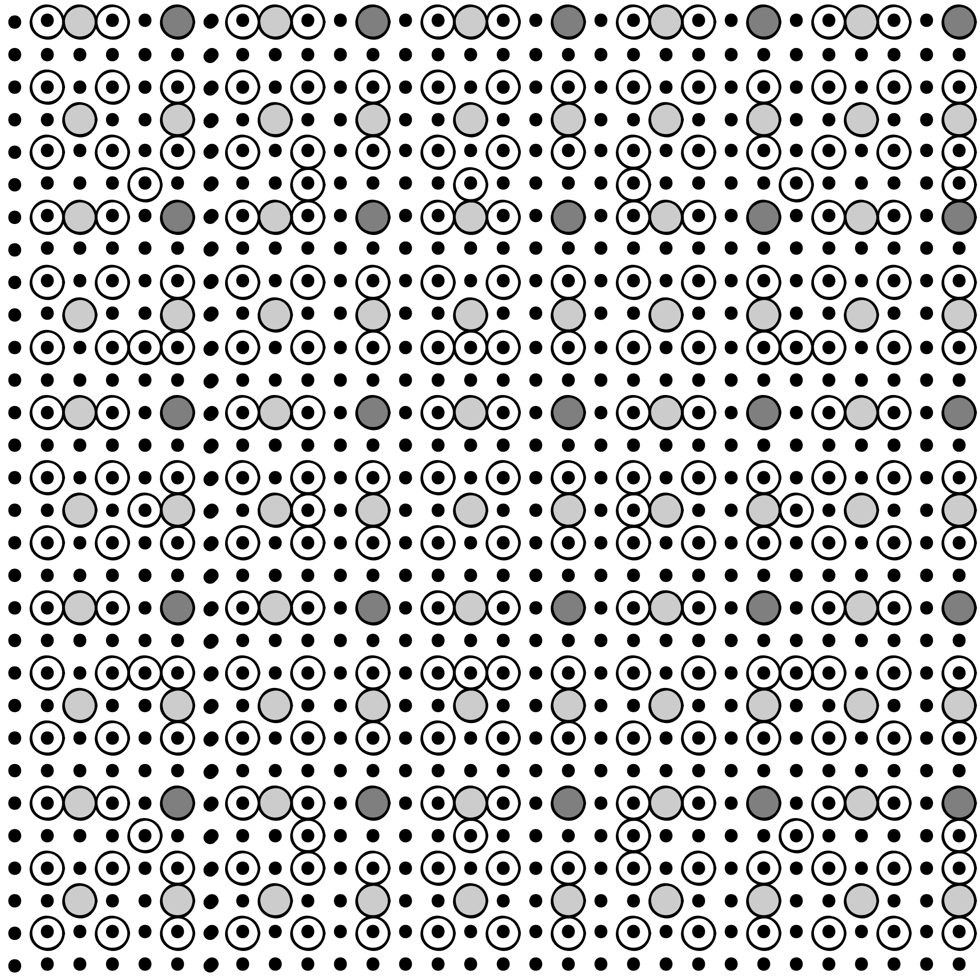

Figure 5: Complete squares in $\Delta_{30,30}$.

externally view $\Delta_{23,20}$.

Problem 5. *Find* $e(23,20)$.

Problem 6. *Find the largest value of* n *such that* $e(n,n) = 3$.

Now that we have a general lower bound for $e(r,s)$, we ask for a general upper bound. Theorem 7 represents our best efforts in this direction, although it involves a sequence, $\delta(n)$, whose terms are not all known. We first define $\rho(i)$ to be the size of the longest sequence of positive integers sharing a common factor with i. Then we define $\delta(n) = \max\{\rho(i) | i \leq n\}$.

Theorem 7. $e(n, n) \leq \delta(n) + 1$.

Proof. By the definition of $\delta(n)$, for any $a < n$, if we order the points in $\Delta_{n,n}$ with x-coordinate a by increasing y-value, then there are at most $\delta(n)$ of these points in sequence not visible from $(0, 0)$.

Therefore we may take the external points $(0, 0)$, $(0, 1)$, $\ldots$, $(0, \delta(n))$. For any point (a, b) in $\Delta_{n,n}$, one of the integers b, $b - 1$, ..., $b - \delta(n)$ is relatively prime to a. Say that $b - k$ is relatively prime to a. Then $(0, k)$ and (a, b) are mutually visible. Therefore the $\delta(n) + 1$ points listed can view all of $\Delta_{n,n}$, and $e(n, n) \leq \delta(n) + 1$. □

Problem 8. *Find a closed formula for* $\delta(n)$*.*

We suspect that Problem 8 is hard, and point the reader towards [8] as a further source of information. However, we do know that $\delta(1) = 1$ and $\delta(2) = 3$, so Theorem 7 gives us $e(5, 5) \leq 2$, which is exact, but $e(6, 6) \leq 4$, which is not.

$e(r, s)$	1	2-5	6-14	15-19	20-29	30	n
1	1	1	1	1	1	1	1
2-5		2	2	2	2	2	2
6-14			3	3	3	3	3
15-23				3	$3-4$	$3-4$	$3-4$
24-29					$3-4$	$3-4$	$3-4$
30						$4-6$	$4-6$

Table 1: The values of $e(r, s)$ for $1 \leq r, s \leq 30$.

We summarize the known values of $e(r, s)$ in Table 1. When a specific value is not known, the range of possible values is given. Note that since

$e(r, s) = e(s, r)$ as remarked above, the table is symmetric, and so values below the diagonal are omitted.

Conjecture 9. *If $r \leq s$ and $r \leq t$ then $e(r, s) = e(r, t)$.*

Problem 10. *Find the values of n for which $e(n, n) > e(n-1, n-1)$.*

Marching Bands in Space

We can make an easy generalization to n-dimensional space from our formal definitions, although the rocket packs required to support a three-dimensional marching band formation are as yet prohibitively expensive, and dimensions larger than three are even worse. As noted above, weak visibility does not become any more interesting in higher dimensions. There are, however, some interesting things we can say about external visibility in higher dimensions.

Analogously, let $\Delta_{r_1, r_2, \ldots, r_n}$ be the n-dimensional box of lattice points with corner at $(1, 1, \ldots, 1)$, and let $e_n(r_1, r_2, \ldots, r_n)$ be the smallest number of points outside of $\Delta_{r_1, r_2, \ldots, r_n}$ required to externally view $\Delta_{r_1, r_2, \ldots, r_n}$. We add the subscript n so that the dimension we're working in is clear. Herzog and Stewart proved this generalization of Theorem 3.

Theorem 11 (Herzog and Stewart, 1971). *A given pattern P is realizable if and only if P fails to contain a complete hypercube modulo p for every prime p.*

Note that in particular, this means that $e_n(2, 2, \ldots, 2) = 2$. However, since $e(1, 2) = 1$, $e_n(1, 2, 2, \ldots, 2) = 1$ for all n. More generally, if r_i and r_j are the smallest of the numbers r_1, r_2, ..., r_n, then $e_n(r_1, r_2, \ldots, r_n) \leq e(r_i, r_j)$, since we can take the points required to externally view Δ_{r_i, r_j} in two dimensions, and add arbitrary additional coordinates to increase their dimension to n, and these new points will externally view $\Delta_{r_1, r_2, \ldots, r_n}$.

So our values of $e(r, s)$ in two dimensions are upper bounds for the values of $e_n(r_1, r_2, \ldots, r_n)$ in n dimensions.

Problem 12. *Find an example of integers r, s, and t for which $r \leq s < t$ and $e(r,s) > e(r,s,t)$, or prove that one does not exist.*

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