Tube Representations of Ordered Sets

Joshua D. Laison (jlaison@coloradocollege.edu)
Department of Mathematics
Colorado College
14 E. Cache La Poudre St.
Colorado Springs, CO 80903

Abstract. We define the (n, i, f)-tube orders, which include interval orders, trapezoid orders, triangle orders, weak orders, order dimension n, and interval-order-dimension n as special cases. We investigate some basic properties of (n, i, f)-tube orders, and begin classifying them by containment.

Keywords: trapezoid order, triangle order, free triangle order

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1. Introduction

Suppose that R is a finite set of closed intervals on the real line (drawn horizontally). If x and y are intervals in R, then we define $x < y$ if and only if x lies entirely to the left of y. We call this order relation the *standard ordering* of R. An ordered set P is called an *interval order* if there exists an ordered set R, whose elements are intervals ordered by the standard ordering, such that $P \cong R$. This isomorphism is called an *interval representation* of P. We also refer to the set R as an interval representation of P.

A number of generalizations of interval orders have been defined. In particular, the trapezoid orders [1, 8] and the triangle orders [4, 7]. In this paper we generalize still further to the (n, i, f)-tube orders, which subsume interval, triangle, and trapezoid orders as special cases. The (n, i, f)-tube orders also specialize to the dimension n orders and the interval-order-dimension n orders.

2. Definitions and Notation

Let $B_1, \ldots B_n$ be parallel lines in n-dimensional space, each parallel to the x_1-axis, and otherwise in general position. In other words, no k of these lines lie in the same $(k-1)$-dimensional space. We define an *n-tube* to be the convex hull of n lines of this form. We call these lines the *baselines* of the n-tube. Note that the canonical ordering on the x_1-axis induces a natural, consistent ordering on each of the baselines.

We will always consider the x_1-axis to be oriented horizontally, so that if x and y are two intervals on a baseline B_k, and $x < y$ in the standard ordering, we say that x is *to the left* of y, and y is *to the right* of x.

Let T be an n-tube, and let X be a set of polytopes contained in T, such that for every polytope $p \in X$, at least one point of p lies on every one of the baselines of T. We call the intersection $p \cap B_k$ the *base* of p on the baseline B_k, and denote it by $\overline{p}_k$. The *standard ordering* of the set X is defined as follows. If p and q are two polytopes in X, then $p < q$ if and only if p and q are disjoint, and $\overline{p}_1$ is to the left of $\overline{q}_1$. Note that if each polytope in X did not intersect every baseline, then this relation might not be an order relation. For example, consider the 2-tube shown in Figure 1, in which $w < x < y < z < w$. A *tube of polytopes* is an ordered pair (T, R), where T is an n-tube, and R is a set of polytopes with the above properties, ordered with the standard ordering.

Suppose (T, R) is a tube of polytopes. Consider a polytope $p \in R$. For each baseline B_k, $\overline{p}_k$ must be a vertex or an interval. The remaining vertices of p are not contained in any baseline of T. We call the vertices of p on the baselines the *bound* vertices of p and the remaining vertices the *free* vertices of p.

Suppose n, i, and f are non-negative integers, and (T, R) is a tube of polytopes. Suppose further that T is an n-tube, every polytope in R intersects at most i baselines in an interval of positive length, and every polytope in R has at most f free vertices. Then (T, R) is called a *tube of* (n, i, f)*-poytopes*. If P is an ordered set, and there exists a tube of (n, i, f)-poytopes (T, R) and an ordered set isomorphism $\phi : P \to R$, then we call P an (n, i, f)*-tube order* and ϕ an (n, i, f)*-tube representation*. In particular, if (T, R) is a tube of (n, i, f)-poytopes, then R is an (n, i, f)-tube order via the identity function. We denote the class of all (n, i, f)-tube orders by $T(n, i, f)$.

Given an (n, i, f)-tube order P, it will be convenient to display an (n, i, f)-tube representation of P as a tube of (n, i, f)-polytopes (T, R), with each polytope p in R labelled with the unique element $\phi^{-1}(p)$ of P given by the (n, i, f)-tube representation $\phi : P \to R$. We will call such a diagram an (n, i, f)*-tube representation* of P and refer to the ordered pair (T, R) as an (n, i, f)*-tube representation* of P when the labelling is clear from context.

If (T, R) is a tube of (n, i, f)-polytopes, then it has baselines $B_1, \ldots, B_n$. For a polytope p in R, if p has two distinct vertices on the baseline B_k, we will denote them by $v_k^-(p)$ and $v_k^+(p)$, meaning the left and right vertices of p on B_k, respectively. If p has one vertex on B_k, we denote it by $v_k(p)$ or by $v_k^-(p)$ or by $v_k^+(p)$. We denote the horizontal coordinates of $v_k^-(p)$, $v_k^+(p)$, and $v_k(p)$ by $x_k^-(p)$, $x_k^+(p)$, and $x_k(p)$, respectively.

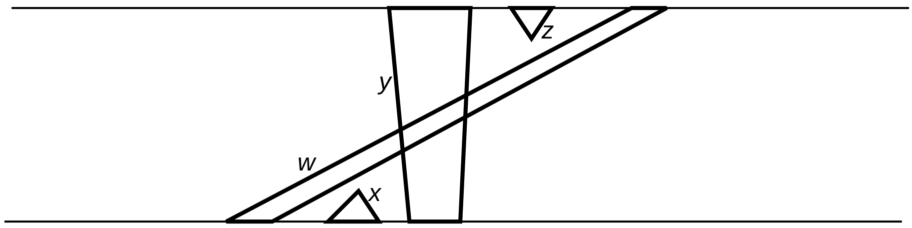


Figure 1. The condition that each polytope intersect every baseline is necessary

Note that since $B_1, \ldots, B_n$ are parallel but otherwise in general position, any two baselines B_k and B_j determine a face of T. We will denote the face of T containing the two baselines B_k and B_j by $f_{kj}(T)$. Since p contains at least one point on B_k and at least one point on B_j, and p is convex, p contains at least a line segment, and possibly a face, in $f_{kj}(T)$. We denote the intersection of p with $f_{kj}(T)$ by $f_{kj}(p)$. We denote by I_k the ordered set of the bases $\overline{p}_k$, $p \in R$, ordered by the standard ordering of intervals on a line.

3. Properties of Tube Representations

We call an (n, i, f)-tube representation (T, R) *pleasant* if it satisfies the following properties:

1. All polytopes in R have exactly f free vertices.
2. All polytopes in R intersect exactly i baselines of T in intervals of positive length.
3. All vertices of all polytopes in R are distinct.
4. All vertices of all polytopes in R have an x_1-value in $\mathbb{N}$.
5. B_1 is the line $\{x_2 = 0, \ldots, x_n = 0\}$, and B_k is the line $\{x_2 = 0, \ldots, x_{k-1} = 0, x_k = 1, x_{k+1} = 0, \ldots, x_n = 0\}$ for $k > 1$.

Lemma 3.1. *If P is an (n, i, f)-tube order, then there exists a pleasant (n, i, f)-tube representation of P.*

Proof. Let P be an (n, i, f)-tube order, and let (T, R) be an (n, i, f)-tube representation of P. We convert (T, R) into a pleasant (n, i, f)-tube representation of P using the following algorithm.

Step 1. Suppose that p is a polytope in R with less than f free vertices.
Choose an edge e of p not on a baseline of T. For any positive

number ϵ, let $p+\epsilon$ be the set of points within distance ϵ of p. We choose an ϵ such that all polytopes other than p either intersect p or are disjoint from $p+\epsilon$. Choose a point x within ϵ of e but not in p. We make x a new free vertex of the polytope p by redefining p to be the convex hull of the old p and the point x. The new polytope p contains every point in the old p and does not intersect any new polytopes, by the choice of x above. Thus the order relation on R is unchanged. We continue this procedure until every polytope in R has exactly f free vertices.

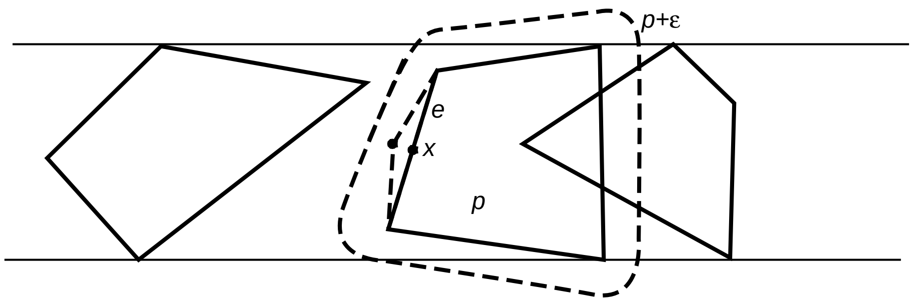


Figure 2. Step 1 of the algorithm

Step 2. Suppose that p is a polytope in R with less than i intervals of positive length on the baselines of T. Choose a baseline B_k such that $\overline{p}_k$ is a single vertex.

Again, we choose an ϵ such that all polytopes other than p either intersect p or are disjoint from $p+\epsilon$. Choose a point x in $p+\epsilon$, on the baseline B_k, but distinct from $\overline{p}_k$. We make x a new vertex of the polytope p by redefining p to be the convex hull of the old p and the point x. We continue this procedure until every polytope in R has exactly i intervals of positive length along the baselines of T. Again, by the choice of x, the order relation on R is unchanged.

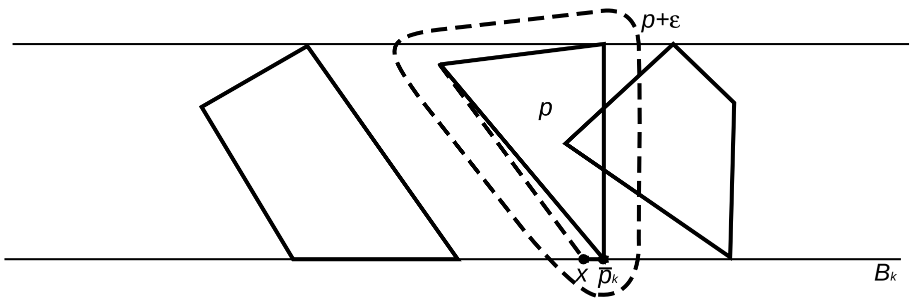


Figure 3. Step 2 of the algorithm

Step 3. Suppose p and q are two polytopes in R that intersect in a single vertex $v_k^+(p) = v_k^-(q)$. Then we define $p+\epsilon$ as above, and choose a point x in $p+\epsilon$, to the right of $v_k^+(p)$ on the baseline B_k. We replace $v_k^+(p)$ with x, and redefine p to be the convex hull of the old p and the point x. We obtain an (n,i,f)-tube representation of P in which $v_k^+(p)$ and $v_k^-(q)$ are distinct.

This procedure also works on if p and q share two left vertices or two right vertices. If p and q share a free vertex, we let L be the line parallel to the baselines of T and containing this vertex, and we replace B_k by L in the procedure above. We continue this procedure until every vertex of every polytope in R is distinct.

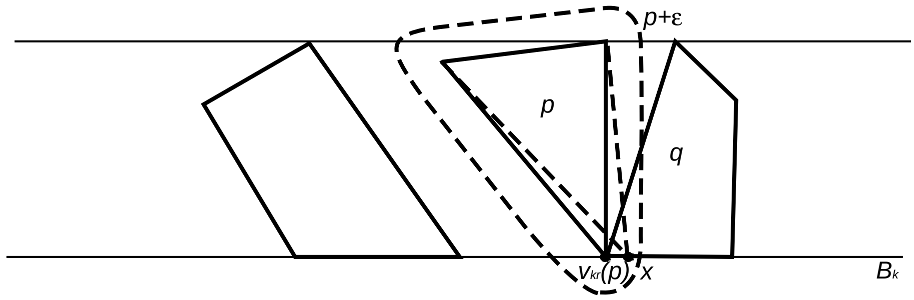


Figure 4. Step 3 of the algorithm

Step 4. We first add a constant value to the x_1-coordinate of every element of R so that every x_1-coordinate of every vertex in R is positive. Then we move every vertex in R horizontally by some small amount so that it has a rational x_1-coordinate, in other words, $x(v) = \frac{a_v}{b_v}$ for all vertices v in R. Let m be the least common multiple of the numbers b_v. We multiply the x_1-coordinates of every polytope in R by m, so that every vertex of every polytope in R has an x_1-coordinate in $\mathbb{N}$.

Step 5. Finally, let H be the hyperplane consisting of all of the points of n-space with x_1-coordinate 0. H is perpendicular to every baseline B_k of T. The intersection of H with T is an $(n-1)$-dimensional simplex s. We perform an affine transformation on n-space which preserves the subspace orthogonal to H and acts on H by moving s to the standard simplex with vertices (in n-space) $(0,\ldots,0)$, $(0,1,\ldots,0)$, $(0,0,1,0,\ldots,0)$, $\ldots$, $(0,\ldots,0,1)$. After this transformation, T has property 5.

The (n,i,f)-tube representation (T,R) is now pleasant. Further, each step of the algorithm did not change the ordering of the polytopes in R. Therefore (T, R) is now a pleasant (n, i, f)-tube representation of P, as claimed. □

Note that every polytope in a pleasant tube of (n, i, f)-polytopes has exactly $n + i + f$ vertices.

An initial inspection of the definitions leads us to the following restrictions on the non-negative integers n, i, and f for any tube of (n, i, f)-polytopes (T, R). The n-tube T must have at least one baseline, so $n > 0$. A polytope can never intersect more baselines in intervals than there are baselines, so $i \leq n$. Finally, if T has only one baseline, then R must contain no free vertices, so if $n = 1$, then $f = 0$. All other combinations of non-negative integers n, i, and f are permissible.

In what follows, we distinguish the intersection of sets in the plane from the intersection of ordered sets common in the theory of ordered sets by calling the latter *order intersection*. The *interval-order dimension* of an ordered set P is the minimal number of interval orders whose order intersection is P.

Lemma 3.2. *Suppose that (T, R) is a tube of (n, i, f)-polytopes.*

1. *If $f = 0$, then R is the order intersection of the interval orders I_k, $1 \leq k \leq n$.*

2. *If $f > 0$, then the order intersection of the interval orders I_k, $1 \leq k \leq n$, is an extension of R.*

Proof. 1. If $f = 0$, then each polytope $p \in R$ is the convex hull of the intervals $\{\overline{p}_k\}_{k=1}^n$. Therefore $p < q$ if and only if $\overline{p}_k$ is to the left of $\overline{q}_k$ for all $1 \leq k \leq n$.

2. If $f > 0$, then the convex hull of the intervals $\{\overline{p}_k\}_{k=1}^n$ is properly contained in p, for all $1 \leq k \leq n$. Therefore $p < q$ implies that $\overline{p}_k$ is to the left of $\overline{q}_k$ for all $1 \leq k \leq n$, but not conversely.

□

Suppose that P is an $(n, 0, 0)$-tube order, and (T, R) is a pleasant $(n, 0, 0)$-tube representation of P. The elements $\{\overline{p}_k\}$ are points, with natural numbers as their x_1-coordinates, and the standard ordering I_k on $\{\overline{p}_k\}$ is obtained by comparing the x_1-coordinates of these elements. In other words, I_k is a weak order, and by Lemma 3.2, $\{R_1, \ldots, R_n\}$ is a weak-order realizer of P. Therefore $T(1, 0, 0)$ is the class of weak orders, and when $n > 1$, $T(n, 0, 0)$ is the class of ordered sets with dimension at most n [8].

Additionally, by Lemma 3.2, the ordered sets in $T(n, n, 0)$ are exactly the class of ordered sets with interval-order-dimension at most n [8].

In particular, $T(1,1,0)$ is the class of interval orders and $T(2,2,0)$ is the class of trapezoid orders [2, 6]. Indeed, the elements in a pleasant $(1,1,0)$-tube representation are intervals, and the elements in a pleasant $(2,2,0)$-tube representation are trapezoids. Similarly, the elements in a pleasant $(2,1,0)$-tube representation are triangles, so $T(2,1,0)$ is the class of triangle orders [7].

In this paper, we are primarily interested in containment relations between classes of (n,i,f)-tube orders. We represent this information succinctly by considering the ordered set $\mathbb{T}$ of classes of (n,i,f)-tube orders, where $T(n_1,i_1,f_1) <_{\mathbb{T}} T(n_2,i_2,f_2)$ if and only if $T(n_1,i_1,f_1) \subseteq T(n_2,i_2,f_2)$. Ryan proved that the interval orders are properly contained in the triangle orders, and the triangle orders are properly contained in the trapezoid orders [7]. Therefore $T(1,1,0) \lneq_{\mathbb{T}} T(2,1,0)$, and $T(2,1,0) \lneq_{\mathbb{T}} T(2,2,0)$. Further, by the remarks above, $T(n_1,0,0) \lneq_{\mathbb{T}} T(n_2,0,0)$ and $T(n_1,n_1,0) \lneq_{\mathbb{T}} T(n_2,n_2,0)$ for $n_1 < n_2$ [8].

Suppose P is an ordered set. If n is the smallest positive integer such that there exist some numbers i and f for which P is an (n,i,f)-tube order, then Habib, Kelly, and Möhring call P an ordered set with *tube dimension* n [3]. They proved that being an ordered set with tube dimension n is a comparability invariant for positive integers n, so the results in this paper hold for the corresponding classes (n,i,f)-tube graphs as well.

4. Containment Relations Between Tube Orders

Lemma 4.1. *If P is an (n_1,i,f)-tube order, then P is an (n_2,i,f)-tube order if $n_1 < n_2$.*

Proof. Let P be an (n_1,i,f)-tube order. There exists a (n_1,i,f)-tube representation (T,R) of P. We consider T as lying in n_2-dimensional space. We form the n_2-tube T' by adding n_2-n_1 baselines to T so that all n_2 baselines are parallel and otherwise in general position in n_2-space. Next, we choose points $\{x_p\}_k$ on each new baseline B_k, indexed by the elements of P, so that the x_1-coordinate of the point x_{p_k} is to the left of the x_1-coordinate of the point x_{q_k} if $p<q$, for all $n_1+1 \leq k \leq n_2$. Form the new polytope p' by taking the convex hull of the points $\{x_p\}$ and p. Let R' be the set of polytopes $\{p'\}$, ordered by the standard ordering. We know that $R' \cong P$ since the ordered set R' is formed by taking the intersection of R with some extensions of R. Therefore (T',R') is an (n_2,i,f)-tube representation of P, and so P is an (n_2,i,f)-tube order. □

Lemma 4.2. *If P is an (n, i_1, f)-tube order, then P is an (n, i_2, f)-tube order if $i_1 < i_2$.*

Proof. Let P be an (n, i_1, f)-tube order. P has an (n, i_1, f)-tube representation, which is also an (n, i_2, f)-tube representation. Therefore P is an (n, i_2, f)-tube order. □

Lemma 4.3. *If P is an (n, i, f_1)-tube order, then P is an (n, i, f_2)-tube order if $f_1 < f_2$.*

Proof. Let P be an (n, i, f_1)-tube order. P has an (n, i, f_1)-tube representation, which is also an (n, i, f_2)-tube representation. Therefore P is an (n, i, f_2)-tube order. □

Lemma 4.4. *If P is an (n_1, i_1, f_1)-tube order, then P is an (n_2, i_2, f_2)-tube order if $n_1 \leq n_2$, $i_1 \leq i_2$, and $f_1 \leq f_2$.*

Proof. Let P be an (n_1, i_1, f_1)-tube order. By Lemma 4.1, P is an (n_2, i_1, f_1)-tube order. By Lemma 4.2, P is an (n_2, i_2, f_1)-tube order. Finally, by Lemma 4.3, P is an (n_2, i_2, f_2)-tube order. □

The next lemma is a generalization of Lemma 4.3.

Lemma 4.5. *If P is an (n, i_1, f_1)-tube order, then P is an (n, i_2, f_2)-tube order if $f_1 \leq f_2$ and $i_1 + f_1 \leq i_2 + f_2$.*

Proof. Let P be an (n, i_1, f_1)-tube order, and let (T, R) be a pleasant (n, i_1, f_1)-tube representation of P. Since (T, R) is pleasant, each polytope in R has a base of positive length on exactly i_1 baselines of T, and has exactly f_1 free vertices.

If $i_1 \leq i_2$ then the statement is true by Lemmas 4.2 and 4.3. Therefore we may assume that $i_1 > i_2$. For each polytope p in R, we choose $i_1 - i_2$ baselines on which p has a base of positive length, and an endpoint of each of those bases. These are $i_1 - i_2$ bound vertices of p.

Let $v_k^-(p)$ be one of these vertices. We choose a positive number ϵ such that all polytopes other than p either contain or are disjoint from a ball B of radius ϵ centered at $v_k^-(p)$. We choose a point x in B and in the interior of T. We then redefine p by replacing $v_k^-(p)$ with the point x, and letting the new p be the convex hull of every other vertex of p and x. The new polytope p still intersects the same set of polytopes in R as it did previously. We repeat this process for every one of the chosen vertices.

Each polytope in R now intersects exactly $i_1 - (i_1 - i_2) = i_2$ baselines of T in intervals, and has exactly $f_1 + i_1 - i_2$ free vertices. Therefore (T, R) is an $(n, i_2, f_1 + i_1 - i_2)$-tube representation of P. Since $f_1 + i_1 - i_2 \leq f_2$ by the hypothesis, (T, R) is also an (n, i_2, f_2)-representation of P. □

Suppose that x is a free vertex of a polytope p in a tube of (n, i, f)-polytopes (T, R). Let L be the unique line parallel to the baselines of T and containing x. We say that x is *left-pointing* if x is the left endpoint of the interval $p \cap L$. We say that p is *left-pointing* if every one of its free vertices is left-pointing. We say that R is *left-pointing* if every one of its polytopes is left-pointing.

In [4], we defined an *asterisk* in a tube of $(2, 2, 0)$-polytopes to be an ordered triple of three polytopes (p, q, r), such that $x_2^+(p) < x_2^+(q) < x_2^+(r)$ and $x_1^+(r) < x_1^+(q) < x_1^+(p)$. We denoted the unique left and right edges of a polygon in a tube of $(2, 2, 0)$-polytopes by $e^-(t)$ and $e^+(t)$, respectively. We proved the following.

Lemma 4.6. *Every $(2, 2, 0)$-tube order P has a $(2, 2, 0)$-tube representation (R, T) that satisfies the following property:*

Property A. *For every asterisk (p, q, r) in R, $e^+(q)$ passes to the left of the intersection of $e^+(p)$ and $e^+(r)$.*

We generalize lemma 4.6 as follows:

Lemma 4.7. *Every $(n, i, 0)$-tube order P has an $(n, i, 0)$-tube representation (R, T) that satisfies the following property:*

Property B. *For every face $f_{ij}(T)$ of T, the induced $(2, 2, 0)$-tube representation on $f_{ij}(T)$ satisfies Property A.*

Proof. In Lemma 4.6, we obtained (R, T) by starting with an arbitrary $(2, 2, 0)$-tube representation of P, and applying the function $\phi(x, y) = (3^x, y)$. Starting with an $(n, i, 0)$-tube representation (R', T') of P, we apply the function $\phi(x_1, \ldots, x_n) = (3^{3^{\cdots 3^{x_1}}}, \ldots, 3^{x_{n-1}}, x_n)$ to T' to obtain a $(n, i, 0)$-tube representation (R, T) of P with Property B. $\square$

We then proved the following.

Theorem 4.8. *If (R, T) is a $(2, 2, 0)$-tube representation of an ordered set P which satisfies Property A, then there exists a left-pointing $(2, 0, 1)$-tube representation (R', T') of P.*

We generalize Theorem 4.8 as follows.

Theorem 4.9. *If P is an $(n, i, 0)$-tube order, then P is an $(n, i - 2, 1)$-tube order if $i \geq 2$.*

Proof. Let (R, T) be an $(n, i, 0)$-tube representation of P satisfying Property B. Since no polytope in R has any free vertices, each polytope in R is determined by its intersection with the faces of T. We choose a face f of T and form the tube of $(2, 2, 0)$-polytopes induced by the faces of the polytopes in R. Since (R, T) satisfies Property B, the induced tube of $(2, 2, 0)$-polytopes in f satisfies Property A, so we can apply Theorem 4.8 to form a tube of $(2, 0, 1)$-polytopes with the same standard ordering. Note that Theorem 4.8 still works even if some of the polygons in f are actually line segments. We then repeat the process with another face of T, but we leave any polytope already modified in the previous face unchanged. Since we can apply Theorem 4.8 to the $(2, 2, 0)$-polytopes in a given face in any order we wish, and the standard ordering on R is the same after each one, we can stop this process whenever we wish. We stop when we have modified a face of every polytope in R exactly once.

We then take the convex hull of each of the new faces to obtain a new set of polytopes R'. Since R' is also determined by its intersection with the faces of T, (R', T) is a tube representation of P. Since each polytope p in R had at least two intervals of positive length along two of the baselines of T, p has exactly two less in R'. Therefore (R', T) is an $(n, i-2, 1)$-tube representation of P.

□

Notice that the $(n, i-2, 1)$-tube representation we obtained in the proof of Theorem 4.9 does not use the full power of an $(n, i-2, 1)$-tube representation, since the free vertices of each polytope still lie on the boundary of the tube. This suggests that we might use the free vertices more efficiently to obtain, for example, an $(n, i-3, 1)$-tube representation of every $(n, i, 0)$-tube order. At the present time, we have not been able to do this.

5. Separating Examples for Distinct Values of n

Now we turn our attention to the cases for which we can prove $T(n_1, i_1, f_1) \not\subseteq T(n_2, i_2, f_2)$. In each case, we build an example that will serve to distinguish two classes of tube orders.

Recall that the ordered set S_n, the standard example of an n-dimensional ordered set, contains the 1-element and $(n-1)$-element subsets of $[n]$, ordered by inclusion. We denote by S_n^+ the lexicographic sum of $\{\mathbf{2}, \ldots, \mathbf{2}\}$ over the ordered set S_n. In other words, we replace every element in S_n with a chain of length 2. We denote an element of S_n^+ by its row and column in the diagram, as shown in Figure 5.

Recall that if (T, R) is a tube of (n, i, f)-polytopes and B_α is a baseline of T, then we denote by I_α the ordered set of the intervals $\{\overline{p}_\alpha\}$, $p \in R$, with the standard ordering.

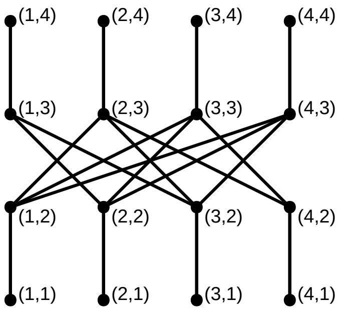


Figure 5. The ordered set S_4^+

Lemma 5.1. *Let (T,R) be an (n,i,f)-tube representation of S_n^+, and reference the elements of R according to Figure 5. Then for each $k \in [n]$, there exists a baseline B_α of T such that $\overline{(k,2)}_\alpha$ is not entirely to the left of $\overline{(k,3)}_\alpha$.*

Proof. Suppose to the contrary that $\overline{(k,2)}_\alpha <_{I_\alpha} \overline{(k,3)}_\alpha$ on every baseline B_α of T. Let x_α be a point in $\overline{(k,2)}_\alpha$ for each $l = 1,\ldots,n$, and let p be the simplex formed by the convex hull of the points $\{x_\alpha\}$. Since $(k,2)$ is convex, $p \subseteq (k,2)$. Since $(k,1) <_R (k,2)$, it follows that $(k,1) <_R p$. Similarly, let y_α be a point in $\overline{(k,3)}_\alpha$ for each $l = 1,\ldots,n$, and let q be the simplex formed by the convex hull of the points $\{y_\alpha\}$. Since $\overline{(k,3)}$ is convex, $q \subseteq (k,3)$, and since $(k,3) <_R (k,4)$, $q <_R (k,4)$.

Since $\overline{(k_2)}_\alpha <_{I_\alpha} \overline{(k,3)}$ on every baseline B_α of R, $p <_R q$, which implies that $(k,1) <_R (k,4)$. But this is a contradiction, since $(k,1) \parallel_R (k,4)$. Therefore there must be some baseline B_α on which $\overline{(k,2)}_\alpha$ is not entirely to the left of $\overline{(k,3)}_\alpha$. □

Lemma 5.2. *Let (T,R) be an (n,i,f)-tube representation of S_n^+, and reference the elements of R according to Figure 5. Then it is not possible that both $\overline{(k,2)}_\alpha < \overline{(k,3)}_\alpha$ and $\overline{(j,2)}_\alpha < \overline{(j,3)}_\alpha$ on the same baseline B_α of T if $k \neq j$.*

Proof. Suppose that $\overline{(k,2)}_\alpha \not<_{I_\alpha} \overline{(k,3)}_\alpha$ and $\overline{(j,2)}_\alpha \not<_{I_\alpha} \overline{(j,3)}_\alpha$ on the same baseline B_α of R. If $\overline{(k,3)}_\alpha <_{I_\alpha} \overline{(k,2)}_\alpha$ then $\overline{(j,2)}_\alpha <_R \overline{(k,3)}_\alpha <_{I_\alpha} \overline{(k,2)}_\alpha <_R \overline{(j,3)}_\alpha \not>_{I_\alpha} \overline{(j,2)}_\alpha$ is a contradiction, and if $\overline{(j,3)}_\alpha <_{I_\alpha} \overline{(j,2)}_\alpha$ then $\overline{(k,2)}_\alpha <_R \overline{(j,3))l} <_{I_\alpha} \overline{(j,2)}_\alpha <_R \overline{(k,3)}_\alpha \not>_{I_\alpha} \overline{(k,2)}_\alpha$ is a contradiction. Therefore $\overline{(k,2)}_\alpha$ overlaps $\overline{(k,3)}_\alpha$ and $\overline{(j,2)}_\alpha$ overlaps $\overline{(j,3)}_\alpha$. But this is also a contradiction, because this implies that the induced suborder of I_α on the elements $\overline{(k,2)}, \overline{(k,3)}_\alpha, \overline{(j,2)}_\alpha$, and $\overline{(j,3)}_\alpha$ forms a $\mathbf{2}+\mathbf{2}$, which is impossible since I_α is an interval order. □

Theorem 5.3. *The ordered set $S^+_{n_2}$ is an (n_2, i, f)-tube order and not an (n_1, i, f)-tube order if $n_1 < n_2$, for all i and f for which (n, i, f)-tube orders are defined.*

Proof. S_{n_2} has dimension n_2. Therefore let $R_1, \dots R_{n_2}$ be a realizer of S_{n_2}. We form the lexicographic sum R'_k of $\{\mathbf{2}, \dots, \mathbf{2}\}$ over R_k for each $k = 1, \dots, n$. The new set of linear orders $R'_1, \dots, R'_{n_2}$ is a realizer of $S^+_{n_2}$, so $S^+_{n_2}$ is at most n_2-dimensional. Since $T(n_2, 0, 0)$ is precisely the class of ordered sets of dimension n_2 or less for $n_2 > 1$ and contains the linear orders when $n_2 = 1$, $S^+_{n_2}$ is an $(n_2, 0, 0)$-tube order. Therefore $S^+_{n_2}$ is an (n_2, i, f)-tube order by Lemmas 4.2 and 4.3.

It remains to show that $S^+_{n_2}$ is not an (n_1, i, f)-tube order. Suppose that (T, R) is an (m, i, f)-tube representation of $S^+_{n_2}$ for some m. Again we use the notation of Figure 5. By Lemma 5.1, for each $k \in [n_2]$, there exists a baseline B_α of T such that $\overline{k,2}_\alpha \not<_{I_\alpha} \overline{k,3}_\alpha$. By Lemma 5.2, this baseline is distinct for each $k \in [n_2]$. Since there are n_2 pairs of elements of this form, it follows that T has at least n_2 baselines. Therefore $S^+_{n_2}$ is not an (n_1, i, f)-tube order if $n_1 < n_2$, for all i and f for which (n, i, f)-tube orders are defined. □

6. Separating Examples for Distinct Values of i

We form the ordered set $Interval(n, i)$ in the following way. $Interval(n, i)$ will be the union of the two disjoint sets A and B, and the additional element x. We start with two copies of S^+_n, denoted A' and B', with every element of A' less than every element of B'. We add the element x, which is less than all of the maximal elements of B', less than $n - i$ elements covered by these elements, greater than all of the minimal elements of A', and incomparable to every other element of A' and B'. For each maximal element b of B', we add an element b' to B' greater than b, greater than the set of elements $\{y | y < b\}$, and incomparable to every other element of $Interval(n, i)$. The set B' together with all of the elements b' forms the set B. Finally, for every minimal element a of A', we add an element a' to A' less than a, less than the set of elements $\{y | a < y\}$, and incomparable to every other element of $Interval(n, i)$. The set A' together with all of the elements a' forms the set A. We denote an element of A or B by its row and column in the diagram of $Interval(n, i)$, as in Figure 6.

Theorem 6.1. *The ordered set $Interval(n, i_2)$ is an $(n, i_2, 0)$-tube order and not an $(n, i_1, 0)$-tube order if $i_1 < i_2$.*

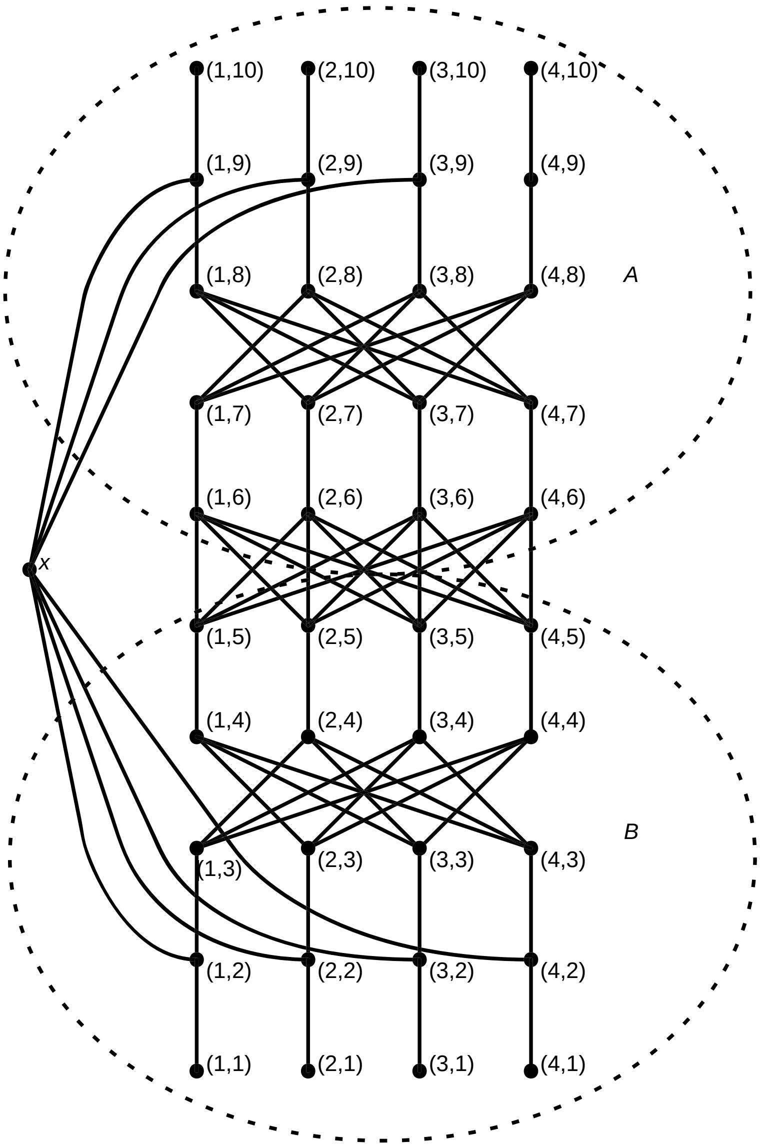


Figure 6. The ordered set $Interval(4,3)$

Proof. First we show that $Interval(n, i_2)$ is an $(n, i_2, 0)$-tube order by constructing an $(n, i_2, 0)$-tube representation (T, R) of $Interval(n, i_2)$. We will do this by defining each of the ordered sets of intervals I_k for each $k = 1 \dots n$. Then we define the polytope p to be the convex hull of the intervals labelled p on each of the n baselines of T.

On the first i_2 baselines, we define I_k to be the interval representation in Figure 7. We use the notation (j, m) to denote the set of intervals $\{(j, m) | j \neq k\}$. These intervals will be identical in I_k, in fact, they will all be the same point. On the remaining baselines, we define I_k to be the interval order in Figure 8, using the same notation. One can check that the ordered set R is isomorphic to the ordered set $Interval(n, i_2)$.

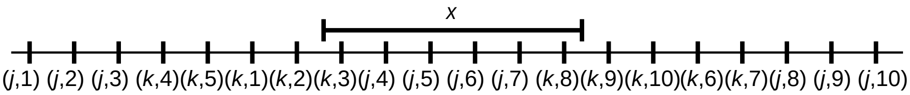


Figure 7. The baseline B_k of the $(n, i_2, 0)$-tube representation (T, R), for the values $1 \leq k \leq i_2$

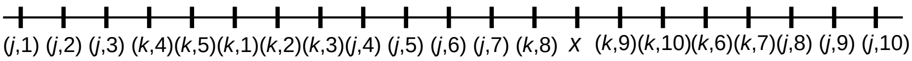


Figure 8. The baseline B_k of the $(n, i_2, 0)$-tube representation (T, R), $i_2 + 1 \leq k \leq n$

Now we will show that $Interval(n, i_2)$ is not an $(n, i_1, 0)$-tube order if $i_1 < i_2$. We show that in any $(n, i, 0)$-tube representation (T, R) of $Interval(n, i_2)$, the polytope labelled x must have two vertices on at least i_2 baselines. This will be sufficient to show that $Interval(n, i_2)$ is not an $(n, i_1, 0)$-tube order if $i_1 < i_2$. We note that in any tube of $(n, i, 0)$-polytopes, if any two polytopes intersect, they must intersect on some baseline.

First, since for each $1 \leq k \leq n$, $x||_R(k, 3)$ and $x||_R(k, 8)$, there is some baseline B_α of R for which $\overline{x}_\alpha$ is not greater than $\overline{(k, 3)}_\alpha$, and some baseline B_β of R for which $\overline{x}_\beta$ is not less than $\overline{(k, 8)}_\beta$. We claim that $\overline{x}_\alpha$ must be less than one of the intervals $\overline{(k, 3)}_\alpha$ and $\overline{(j, 3)}_\alpha$ if $k \neq j$.

For each $1 \leq k \leq n$, there is some baseline B_α for which $\overline{(k, 1)}_\alpha$ is not less than $\overline{(k, 5)}_\alpha$. If $\overline{(k, 1)}_\alpha$ intersects $\overline{(k, 5)}_\alpha$ and $\overline{(k, 4)}_\alpha$ also intersects $\overline{(k, 2)}_\alpha$, then the induced suborder of I_α on the set $\{\overline{(k, 1)}_\alpha, \overline{(k, 2)}_\alpha, \overline{(k, 4)}_\alpha, \overline{(k, 5)}_\alpha\}$ would be a $\mathbf{2} + \mathbf{2}$, which is impossible since I_α is an interval order. On the other hand, if $\overline{(k, 1)}_\alpha$ is greater than $\overline{(k, 5)}_\alpha$, then $\overline{(k, 2)}_\alpha$ must also be greater than $\overline{(k, 4)}_\alpha$. Therefore, $\overline{(k, 4)}_\alpha$ must be less than $\overline{(k, 2)}_\alpha$.

Therefore if $j \neq k$ we have $\overline{(j, 3)}_\alpha < \overline{(k, 4)}_\alpha < \overline{(k, 2)}_\alpha < \overline{x}_\alpha$, which implies that $\overline{(j, 3)}_\alpha < \overline{x}_\alpha$ on B_α. Since there are $n - 1$ baselines of this form, $\overline{(j, 3)}_\alpha$ must not be less than $\overline{x}_\alpha$ on the same baseline B_α on which $\overline{(j, 1)}_\alpha$ is not less than $\overline{(j, 5)}_\alpha$.

However, we cannot have both $\overline{(k, 1)}_\alpha \not< \overline{(k, 5)}_\alpha$ and $\overline{(j, 1)}_\alpha \not< \overline{(j, 5)}_\alpha$ on the same baseline B_α by Lemma 5.2. Therefore $\overline{(k, 3)}_\alpha \not< \overline{x}_\alpha$ for some k on each baseline B_α, for all $1 \le a \le n$. By a similar argument, $\overline{x}_\beta \not< \overline{(k,8)}_\beta$ for at least i_2 distinct values of b. Therefore $\overline{(k,3)}_\alpha \not< \overline{x}_\alpha \not< \overline{(k,8)}_\alpha$ for at least i_2 baselines of T, and since $(k,3) <_R (k,8)$, x must intersect these baselines in an interval of positive length. Therefore x has two vertices on at least i_2 baselines of T. □

7. Separating Examples for Distinct Values of f

We form the ordered set $Free(f,m)$ as the union of the three disjoint ordered sets A, B, and C. A and B are both antichains, the sizes of which depend on the size of C, as described below. C is the (parallel) sum of m copies of **2**, denoted $c(k,1)$ and $c(k,2)$, for $1 \le k \le m$. For each of these 2-chains, there is one element b_k of B incomparable to $c(k,1)$ and $c(k,2)$, and less than every other element in C. For each subset S of size f of $[m]$, there is one element of A incomparable to $\{c(k,1), c(k,2) | k \in S\}$, and less than every other element in C. Finally, every element of A is incomparable to every element of B.

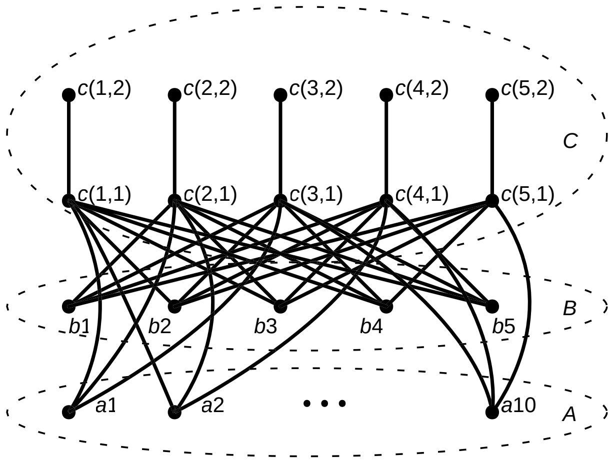


Figure 9. The ordered set $Free(2,5)$

We call a polytope p in a tube of (n,i,f)-polytopes (T,R) *left-visible* (respectively, *right-visible*) with respect to a set of polytopes S in R if $p - S$ is nonempty and there exists a line L contained in T and parallel to its baselines on which $s \not<_L p - S$ for all $s \in S$ (respectively, $s \not>_L p - S$ for all $s \in S$). More generally, a subset s of T is *left-visible* (respectively, *right-visible*) with respect to a subset t of T if $s - t$ is nonempty and there exists a line L contained in T and parallel to its baselines on which $t \not<_L s - t$ (respectively, $t \not>_L s - t$).

Lemma 7.1. *The ordered set $Free(f, m)$ is a $(2, 0, f)$-tube order for all $m > 0$.*

Proof. We construct a $(2, 0, f)$-tube representation (T, R) of $Free(f, m)$. We start by placing the polygons in C so that every polygon of the form $c(k, 1)$ in C is left-visible with respect to $C - \{c(k, 1)\}$, and every polygon of the form $c(k, 2)$ in C is left-visible with respect to $C - \{c(k, 1), c(k, 2)\}$. We do this by letting the polygons $c(k, 1)$ and $c(k, 2)$ be parallel line segments, as in Figure 10.

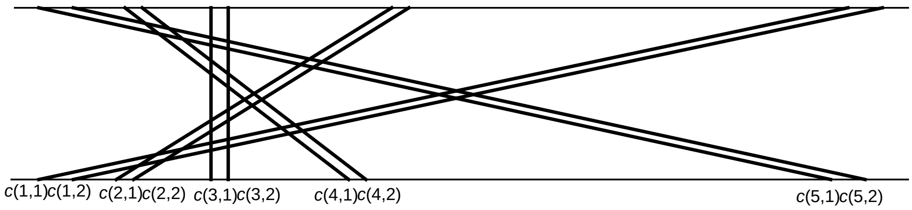


Figure 10. The elements of C in the $(2, 0, f)$-tube representation (T, R)

We now place the polygons in A and B. The polygons in B should each intersect exactly one pair of polygons $c(k, 1)$ and $c(k, 2)$ in C, and they have $f > 0$ free vertices. Since $c(k, 1)$ and $c(k, 2)$ are left-visible with respect to the other elements of C, we place the element of B so that they intersect the required two elements of C in the regions $c(k, 1) - \{c(j, 1), c(j, 2)|j \neq k\}$ and $c(k, 2) - \{c(j, 1), c(j, 2)|j \neq k\}$.

Since the polygons in A should each intersect f elements of C, and each polygon in A has f free vertices, we place these polygons so that they intersect the required elements of C by using one free vertex for each pair of parallel lines $c(k, 1)$ and $c(k, 2)$, as in Figure 11. The convexity of the region to the left of the set $\{c(k, 2)|1 \leq k \leq m\}$ and the fact that we can choose the distances between the parallel lines $c(k, 1)$ and $c(k, 2)$ arbitrarily guarantees that we may place the free vertices of the polygons in A and B so that we get exactly the intersections we want.

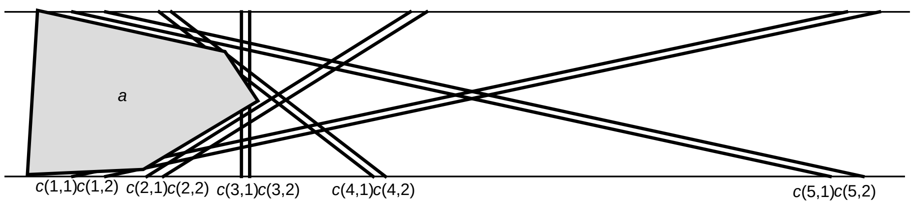


Figure 11. Placing an arbitrary element of A

□

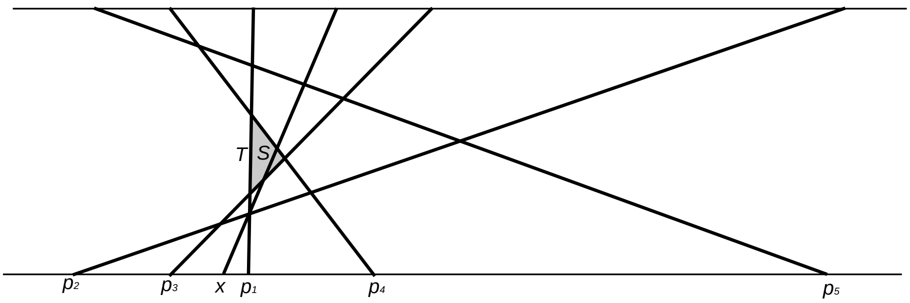


Figure 12. The region S for $n = 2$

Lemma 7.2. *Suppose that* (T, R) *is an* (n, i, f)*-tube representation, and* $p_1, \ldots, p_k$ *are elements of* R *such that* p_j *is left-visible with respect to* $R - p_j$*, for all* $1 \leq j \leq k$*. Let* $S = \{x | x \in T, x \geq p_1, x \leq p_2, \ldots, p_k\}$ *and* $U = \{x | x \in p_1, x \leq p_2, \ldots, p_k\}$*. Then* $p_m \cap S \neq \emptyset$ *implies* $p_m \cap U \neq \emptyset$*, for all* $1 \leq m \leq k$*.*

Proof. Suppose that $p_m \cap S \neq \emptyset$ but $p_m \cap U = \emptyset$, as in Figure 12. Then $p_1 \cap S < p_m \cap S$, by the definition of S. Let $V = \{x | x \in T, x \leq p_2, \ldots, p_k\}$. Then $p_m \cap V$ contains $p_m \cap S$. In fact, since p_m and V are both convex, and p_m does not intersect U, which is the boundary between S and $V - S$, we must have $p_m \cap V = p_m \cap S$. But since $p_1 \cap S < p_m \cap S$, we must have $p_1 < p_m \cap V$, and therefore p_m is not left-visible with respect to $R - p_m$. This contradicts our assumption that p_m is one of the elements $p_1, \ldots, p_k$. Therefore $p_m \cap S \neq \emptyset$ implies $p_m \cap U \neq \emptyset$, for all $1 \leq m \leq k$. □

Theorem 7.3. *The ordered set* $Free(f_2, 2f_2 + 1)$ *is a* $(2, 0, f_2)$*-tube order and not a* $(2, 2, f_1)$*-tube order if* $f_1 < f_2$*.*

Proof. By Lemma 7.1, $Free(f_2, 2f_2 + 1)$ is a $(2, 0, f_2)$-tube order. Suppose that $f_1 < f_2$, and that (T, R) is a $(2, 2, f_1)$-tube representation of $Free(f_2, 2f_2 + 1)$. Let $c(k, 1)$ and $c(k, 2)$ be a comparable pair of C. Let l_k be the line segment connecting $v_{1l}(c(k, 1))$ and $v_{2l}(c(k, 1))$ in R. Since $l_k \subset c(k, 1)$, we know that $l_k < c(k, 2)$. Since b_k intersects $c(k, 2)$, but is less than every polytope in the set $\{c(j, 1) | j \neq k\}$, it follows that b_k intersects l_k, but $b_k < l_j$ for all $j \neq k$. Therefore l_k must be left-visible with respect to the set $\{l_j | j \neq k\}$.

Since the set $\{x | x \leq l_j, j \neq k\}$ is a convex region, the set $s_k = \{x | x \in l_k, x \leq l_j, j \neq k\}$ is a line segment. Let $s = \bigcup s_k$. Since s_k ends where another line segment l_j crosses l_k, s is a piecewise-linear curve with endpoints on B_1 and B_2. Note that every horizontal line L in T intersects s exactly once. Note also that $s < c(k, 2)$ for all $1 \leq k \leq 2f_2 + 1$.

For ease of reference, we renumber $\{c(k,1)\}$, $\{c(k,2)\}$, $\{l_k\}$ and $\{s_k\}$ in the order that the s_k's appear in s, starting with the line segment intersecting B_1 and ending with the line segment intersecting B_2. After this renumbering, there exists an element $a \in A$ such that $a||c(k,1)$ and $a||c(k,2)$ if and only if k is even. In other words, a must not intersect l_1, intersect l_2, not intersect l_3, intersect l_4, and so on. Since the only points in l_k that are left-visible with respect to l_j, $j \neq k$, are contained in s_k, a intersects s_k if and only if k is even.

Since a cannot contain the endpoints of these s_k's, a must have a free vertex to the right of each s_k for which k is even. However, there are f_2 such line segments, and a has only f_1 free vertices. We have obtained a contradiction.

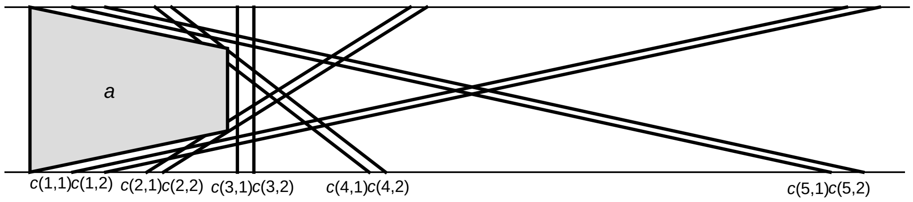


Figure 13. Placing the element a

□

Recall that the *Ramsey number* $R(k,j)$ is the smallest number of vertices for which every graph on $R(k,j)$ vertices has either a clique of size k or an independent set of size j [5]. In particular, since no graph containing a clique of size 5 is planar, every planar graph with $R(5,k)$ vertices has an independent set of size k.

Theorem 7.4. *The ordered set* $Free(f_2, R(5,f_2))$ *is a* $(3,0,f_2)$*-tube order and not a* $(3,3,f_1)$*-tube order if* $f_1 < f_2$.

Proof. The ordered set $Free(f_2, R(5,f_2))$ is a $(2,0,f_2)$-tube order by Lemma 7.1. Therefore the ordered set $Free(f_2, R(5,f_2))$ is a $(3,0,f_2)$-tube order by Lemma 4.1.

Suppose that $f_1 < f_2$ and (T,R) is a $(3,3,f_1)$-tube representation of $Free(f_2, R(5,f_2))$. Let $c(k,1)$ and $c(k,2)$ be a comparable pair in C. Let t_k be the triangle in R with vertices $v_{1,l}(c(k,1))$, $v_{2,l}(c(k,1))$, and $v_{3,l}(c(k,1))$. Since $t_k \subset c(k,1)$, we know that $t_k < c(k,2)$. Since b_k intersects $c(k,2)$, but is less than every polytope in the set $\{c(j,1)|j \neq k\}$, it follows that b_k intersects t_k, but $b_k < t_j$ for all $j \neq k$. Therefore t_k must be left-visible with respect to the set $\{t_j|j \neq k\}$.

Since the set $\{x|x \leq t_j, j \neq k\}$ is a convex region, the set $s_k = \{x|x \in t_k, x \leq t_j, j \neq k\}$ is a convex polygon. Let $s = \bigcup s_k$. Since the edges of s_k occur at the intersections of s_k with another convex polygon s_j, s is a piecewise-linear surface. Again, every horizontal line L in T intersects s exactly once. Note also that $s < c(k,2)$ for all $1 \leq k \leq R(5, f_2)$.

We form a graph G with the polygons s_k as vertices. We include the edge $\{s_k, s_j\}$ in G if and only if s_k intersects s_j. The graph G is planar and has $R(5, f_2)$ vertices. Therefore, it has an independent set I of size f_2.

For ease of reference, we renumber $\{c(k,1)\}$, $\{c(k,2)\}$, $\{l_k\}$ and $\{s_k\}$ so that $I = \{s_1, \ldots, s_{f_2}\}$. After this renumbering, there exists an element $a \in A$ such that $a||c(k,1)$ and $a||c(k,2)$ if and only if $1 \leq k \leq f_2$. Since the only points in t_k that are left-visible with respect to t_j, $j \neq k$, are contained in s_k, a intersects s_k if and only if $1 \leq k \leq f_2$.

Since a cannot contain the boundaries of these s_k's, a must have a free vertex to the right of each s_k for which $1 \leq k \leq f_2$. However, there are f_2 such triangles, and a has only f_1 free vertices. We have obtained a contradiction. □

The technique we have used to show that $Free(f_2, 2f_2 + 1)$ is not a $(2,2,f_1)$-tube order for $f_1 < f_2$ and that $Free(f_2, R(5, f_2))$ is not a $(3,3,f_1)$-tube order for $f_1 < f_2$ does not carry over into four dimensions. In the analogous situation in a 4-tube, the objects s_k that appear in the proofs of Theorem 7.3 and Theorem 7.4 are convex polyhedra filling the interior of a tetrahedron. In dimensions two and three, we were able to claim that the objects s_k were separated by some space, requiring a different free vertex of a to overlap each one. In the four-dimensional case, it is possible that the convex polyhedra are pairwise adjacent no matter how many of them there are. Therefore, a new technique is needed if we wish to prove $T(4, i, f_2) \not\subseteq T(4, i, f_1)$ for $f_1 < f_2$.

8. Separating Examples Involving Both i and f

The following example separates two classes of tube orders by their values of i, but uses techniques from the last two sections.

We form the ordered set $FreeInt(n, i, f, m)$ as the union of three disjoint ordered sets X, Y, and Z. X is isomorphic to the subset A of $Interval(n, i)$ from Section 6. Y is isomorphic to the ordered set $Free(f, m)$ from Section 7. Thus, Y is the union of the sets A, B, and C of $Free(f, m)$. When we refer to the subsets A, B, and C of $FreeInt(n, i, f, m)$, we will mean these subsets of Y. Z is isomorphic to the subset B of $Interval(n, i)$ from Example 6. We use the notation $x(k, j)$, $y(k, j)$, and $z(k, j)$ to reference elements of X, Y, and Z, according to their row and column in the diagram, as in Figure 14.

Every element of B and C is incomparable to every element of Z. Every element of B and C is greater than every element of X. Every element of X is less than every element of Z. Finally, every element of A is less than the elements $\{z(k,10), z(k,9)|k = 1,\ldots,n\}$ and $\{z(j,8)|j = i{+}1,\ldots,n\}$ of Z, greater than the elements $\{x(k,1), x(k,2)|k = 1,\ldots,n\}$ in X, and incomparable to the remaining elements of X and Z.

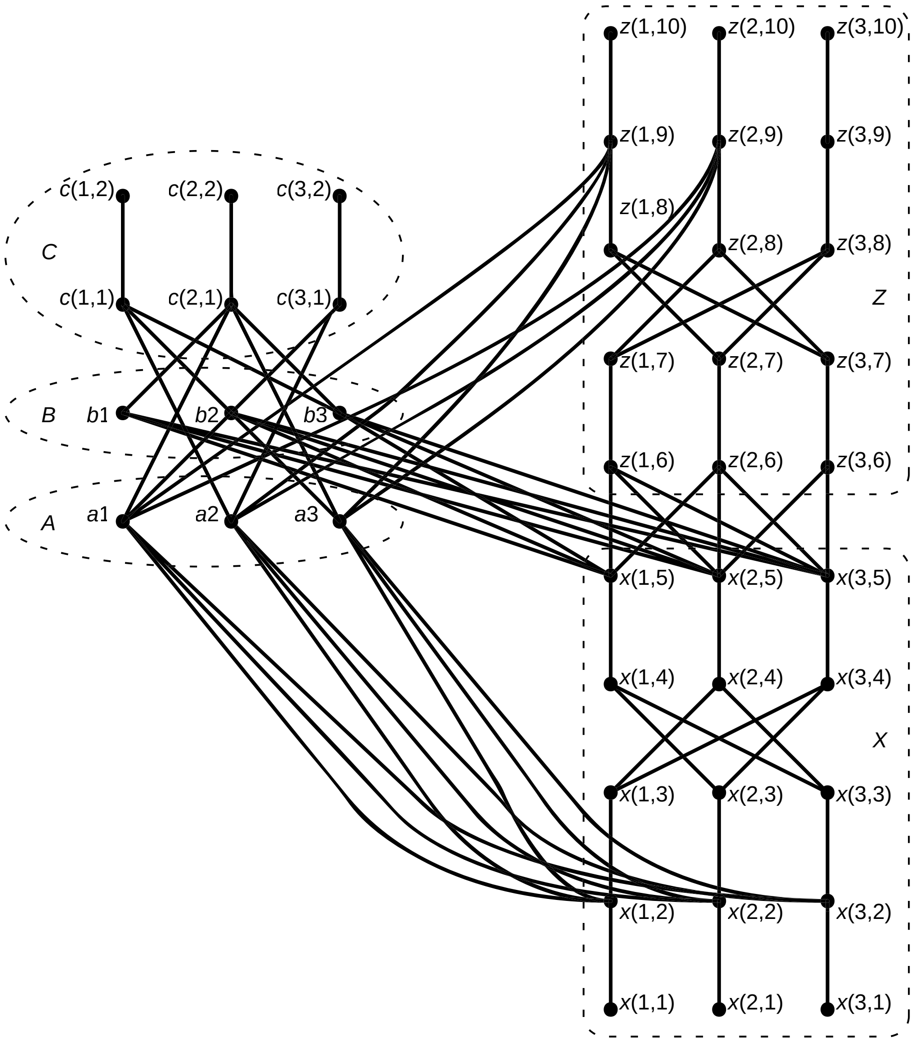


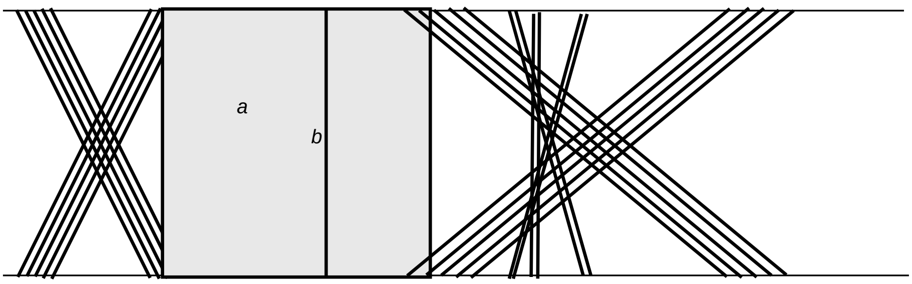
Figure 14. The ordered set $FreeInt(3,2,1,3)$

Theorem 8.1. *The ordered set $FreeInt(2, i_2, f, 2f+1)$ is a $(2, i_2, f)$-tube order and not a $(2, i_1, f)$-tube order if $i_1 < i_2$.*

Proof. Note that $X \cup a \cup Z \cong Interval(2, i_2)$ for each $a \in A$. By Theorem 6.1, let (R_0, T) be a $(2, i_2, 0)$-tube representation of $X \cup a \cup Z$ for some $a \in A$. We then let (R_1, T) be the $(2, i_2, 0)$-tube representation of $X \cup A \cup Z$ obtained from R_0 by including $\binom{2f+1}{f}$ identical copies of the polygon a, one for each element of A. Since the elements of A are pairwise incomparable, $R_1 \cong X \cup A \cup Z$.

Note also that $A \cup B \cup C \cong Free(f, 2f + 1)$. By Theorem 7.3, let (R_2, T) be a $(2, 0, f)$-tube representation of $A \cup B \cup C$. Without loss of generality, we may choose R_2 so that $b, c \cap z \neq \emptyset$ for all $c \in C \subset R_2$, $b \in B \subset S$, and $z \in Z \subset R_1$, and $b, c \cap x = \emptyset$ for all $c \in C \subset R_2$, $b \in B \subset S$, and $x \in X \subset R_1$.

Now for each $a \in A$, let a' be the convex hull of the polytope $a \in R_1$ and the free vertices of the polytope $a \in R_2$. Let $R = (R_1 - A) \cup (R_2 - A) \cup \{a'\}$. We claim that $R \cong FreeInt(2, i_2, f, 2f + 1)$. Since $R_1 \cong X \cup A \cup Z$ and $R_2 \cong A \cup B \cup C$, we need only check the relations between B and X, B and Z, C and X, and C and Z. But by our choice of R_2, these relations agree with $FreeInt(2, i_2, f, 2f + 1)$ as well.



X Z C

Figure 15. The $(2, i_2, 0)$-tube representation R_2

We now show that $FreeInt(2, i_2, f, 2f + 1)$ is not a $(2, i_1, f)$-tube order if $i_1 < i_2$. Assume that (T, R) is a $(2, i_1, f)$-tube representation of $FreeInt(2, i_2, f, 2f + 1)$, $i_1 < i_2$. By Theorem 7.3, there exists an element $a \in A$ and elements $c(1, i_1), \ldots, c(1, i_f)$ such that each of the free vertices $v_{f_1}(a), \ldots, v_{f_f}(a)$ of a intersects a distinct polygon $c(1, i_k)$, $1 \leq k \leq f$. Each of these vertices is therefore greater than every polygon in X. Also, each of these vertices is less than every polygon in Z by Lemma 7.2. Therefore for every element $x \in X \cup Z$ for which $a||x$, $a \cap x$ contains points in B_1 or B_2. However, as we have noted, $X \cup a \cup Z \cong Interval(n, i)$. Therefore by Theorem 6.1, a contains an interval of positive length on at least i_2 of the baselines B_1 and B_2. This contradicts our assumption that $i_1 > i_2$. □

Theorem 8.2. *The ordered set* $FreeInt(3, i_2, f, R(5, f))$ *is a* $(3, i_2, f)$*-tube order and not a* $(3, i_1, f)$*-tube order if* $i_1 < i_2$.

Proof. We construct a $(3, i_2, f)$-tube representation (T, R) of $FreeInt(3, i_2, f, R(5, f))$ as follows. By Theorem 8.1, let (R_0, T_0) be a $(2, i_2, f)$-tube representation of $FreeInt(2, i_2, f, R(5, f))$. Let (I_3, B_3) be the $(1, 1, 0)$-tube representation shown in Figure 16 and (I_3', B_3) be the $(1, 1, 0)$-tube representation shown in Figure 17. Let B_3 be parallel but disjoint from R_0, and let T be the 3-tube formed by taking the convex hull of B_3 and R_0. If $i_2 = 3$, let R be the $(3, i_2, f)$-tube representation formed by taking the convex hull of each polytope in R_0 with its corresponding line segment in I_3. If $i_2 \in \{0, 1, 2\}$, let R be the $(3, i_2, f)$-tube representation formed by taking the convex hull of each polytope in R_0 with its corresponding line segment in I_3'.

One can check that R is a $(3, i_2, f)$-tube representation of $FreeInt(3, i_2, f, R(5, f))$.

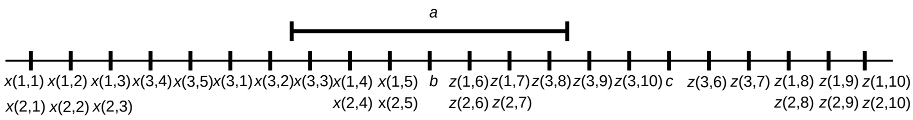


Figure 16. The baseline B_3 of the $(3, 3, f)$-tube representation R

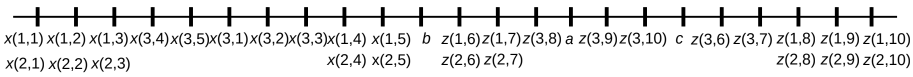


Figure 17. The baseline B_3 of the $(3, i_2, f)$-tube representation R, $i_2 = 0, 1, 2$

The proof that $FreeInt(3, i_2, f, R(5, f))$ is not a $(3, i_1, f)$-tube order if $i_1 < i_2$, is identical to the corresponding part of the two-dimensional case in Theorem 8.1. □

9. Separating Examples Involving Distinct Values of n, i, and f

Lemma 9.1. *The ordered set S_k from Section 5 is an (n, i, f)-tube order if $n > 1$ and $f > 0$ and not an $(n, i, 0)$-tube order if $n < k$.*

Proof. A $(2, 0, 1)$-tube representation of S_k is shown in Figure 18. Therefore S_k is a $(2, 0, 1)$-tube order, and therefore S_k is an (n, i, f)-tube order if $n > 1$ and $f > 0$ by Lemmas 4.1, 4.2 and 4.3.

It is known that S_k has interval-order-dimension k [8]. Therefore S_k is not a $(k-1, k-1, 0)$-tube order. Since $n < k$ implies $i < k$, S_k is not an $(n, i, 0)$-tube order if $n < k$ by Lemmas 4.1 and 4.2. □

Example 9.2. *Consider the set of all closed intervals with endpoints in $[k]$. We will denote the ordered set with this $(1, 1, 0)$-tube representation by $CI(k)$, and call it the canonical interval order of size k.*

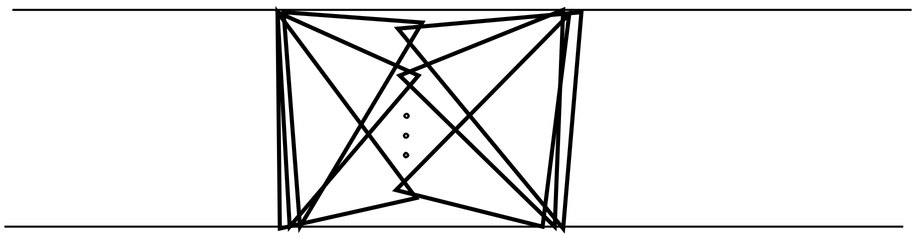


Figure 18. A $(2,0,1)$-tube representation of S_k

The following theorem appears in [8].

Theorem 9.3. *The dimension of $CI(k)$ is at least $log_2 log_2 k + (\frac{1}{2} + o(1)) log_2 log_2 log_2 k$.*

Corollary 9.4. *The ordered set $CI(k)$ is an (n,i,f)-tube order if $i > 0$ or $f > 0$, and not an $(n,0,0)$-tube order for sufficiently large k.*

Proof. By definition, $CI(k)$ is a $(1,1,0)$-tube order, and therefore also a $(2,0,1)$-tube order by Lemmas 4.1 and 4.5. By Lemmas 4.1, 4.2 and 4.3, this implies that $CI(k)$ is also an (n,i,f)-tube order if $i > 0$ or $f > 0$.

On the other hand, given a positive integer n, we can choose k such that $CI(k)$ has dimension greater than n by Theorem 9.3. Therefore $CI(k)$ is not an $(n,0,0)$-tube order for this choice of k. □

10. Summary

We have described every containment and non-containment currently known between classes of (n,i,f)-tube orders. We represent this information succinctly by considering the ordered set $\mathbb{T}$ of classes of (n,i,f)-tube orders, where $T(n_1,i_1,f_1) <_{\mathbb{T}} T(n_2,i_2,f_2)$ if and only if $T(n_1,i_1,f_1) \subseteq T(n_2,i_2,f_2)$.

The following is a table of relations in the ordered set $\mathbb{T}$, together with the theorem or lemma that implies that relation. All classes of tube orders with $n \leq 3$ and $i+f \leq 3$ are listed. Note that each relation requires two theorems, so any entry in the table listing only one theorem or lemma is open with regards to the other direction.

	$T(1,1,0)$	$T(2,0,0)$	$T(2,1,0)$	$T(2,2,0)$
$T(1,0,0)$	$\lneq$, L4.4, T6.1	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(1,1,0)$		$\|\|$, T5.3, C9.4	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,0,0)$			$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T6.1
$T(2,1,0)$				$\lneq$, L4.4, T6.1

	$T(2,0,1)$	$T(2,1,1)$	$T(2,2,1)$	$T(2,0,2)$
$T(1,0,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(1,1,0)$	$\lneq$, L4.5, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.5, L4.4, T5.3
$T(2,0,0)$	$\lneq$, L4.4, T7.3	$\lneq$, L4.4, T7.3	$\lneq$, L4.4, T7.3	$\lneq$, L4.4, T7.3
$T(2,1,0)$	$\lneq$, L4.5, T7.3	$\lneq$, L4.4, T7.3	$\lneq$, L4.4, T7.3	$\lneq$, L4.5, T7.3
$T(2,2,0)$	$\lneq$, T4.8, T7.3	$\lneq$, L4.5, T7.3	$\lneq$, L4.4, T7.3	$\lneq$, L4.5, T7.3
$T(2,0,1)$		$\lneq$, L4.4, T7.3	$\lneq$, L4.4, T7.3	$\lneq$, L4.5, T7.3
$T(2,1,1)$			$\lneq$, L4.4, T8.1	$\lneq$, L4.5, T7.3
$T(2,2,1)$				$\not>$, T7.3

	$T(2,1,2)$	$T(3,0,0)$	$T(3,1,0)$	$T(3,2,0)$
$T(1,0,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(1,1,0)$	$\lneq$, L4.4, T5.3	$\|\|$, T5.3, C9.4	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,0,0)$	$\lneq$, L4.4, T7.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,1,0)$	$\lneq$, L4.4, T7.3	$\|\|$, T5.3, C9.4	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,2,0)$	$\lneq$, L4.5, T7.3	$\|\|$, T5.3, C9.4	$\not>$, T5.3	$\lneq$, L4.4, T5.3
$T(2,0,1)$	$\lneq$, L4.4, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3
$T(2,1,1)$	$\lneq$, L4.4, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3
$T(2,2,1)$	$\lneq$, L4.5, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3
$T(2,0,2)$	$\lneq$, L4.4, T8.1	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3
$T(2,1,2)$		$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3	$\|\|$, T5.3, T7.3
$T(3,0,0)$			$\lneq$, L4.4, T6.1	$\lneq$, L4.4, T6.1
$T(3,1,0)$				$\lneq$, L4.4, T6.1

	$T(3,3,0)$	$T(3,0,1)$	$T(3,1,1)$
$T(1,0,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(1,1,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, L4.5, T5.3	$\lneq$, L4.4, T5.3
$T(2,0,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,1,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, L4.5, T5.3	$\lneq$, L4.4, T5.3
$T(2,2,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T4.8, T5.3	$\lneq$, L4.4, T4.5, T5.3
$T(2,0,1)$	$\|\|$, T5.3, T7.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,1,1)$	$\|\|$, T5.3, T7.3	$\not>$, T5.3	$\lneq$, L4.4, T5.3
$T(2,2,1)$	$\|\|$, T5.3, T7.3	$\not>$, T5.3	$\not>$, T5.3
$T(2,0,2)$	$\|\|$, T5.3, T7.3	$\not>$, T5.3	$\not>$, T5.3
$T(2,1,2)$	$\|\|$, T5.3, T7.3	$\not>$, T5.3	$\not>$, T5.3
$T(3,0,0)$	$\lneq$, L4.4, T6.1	$\lneq$, L4.4, T7.4	$\lneq$, L4.4, T7.4
$T(3,1,0)$	$\lneq$, L4.4, T6.1	$\lneq$, L4.5, T7.4	$\lneq$, L4.4, T7.4
$T(3,2,0)$	$\lneq$, L4.4, T6.1	$\lneq$, T4.9, T7.4	$\lneq$, T4.9, L4.4, T7.4
$T(3,3,0)$		$\not>$, T7.4	$\lneq$, T4.9, T7.4
$T(3,0,1)$			$\lneq$, L4.4, T8.2

	$T(3,2,1)$	$T(3,0,2)$	$T(3,1,2)$
$T(1,0,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(1,1,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, L4.5, T5.3	$\lneq$, L4.4, T5.3
$T(2,0,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,1,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, L4.5, T5.3	$\lneq$, L4.4, T5.3
$T(2,2,0)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, L4.5, T5.3	$\lneq$, L4.4, L4.5, T5.3
$T(2,0,1)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,1,1)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, L4.5, T5.3	$\lneq$, L4.4, T5.3
$T(2,2,1)$	$\lneq$, L4.4, T5.3	$\lneq$, L4.5, L4.4, T5.3	$\not>$, T5.3
$T(2,0,2)$	$\not>$, T5.3	$\lneq$, L4.4, T5.3	$\lneq$, L4.4, T5.3
$T(2,1,2)$	$\not>$, T5.3	$\not>$, T5.3	$\lneq$, L4.4, T5.3
$T(3,0,0)$	$\lneq$, L4.4, T7.4	$\lneq$, L4.4, T7.4	$\lneq$, L4.4, T7.4
$T(3,1,0)$	$\lneq$, L4.4, T7.4	$\lneq$, L4.5, T7.4	$\lneq$, L4.4, T7.4
$T(3,2,0)$	$\lneq$, L4.4, T7.4	$\lneq$, T4.5, T7.4	$\lneq$, T4.5, L4.4, T7.4
$T(3,3,0)$	$\lneq$, T4.9, L4.4, T7.4	$\lneq$, T4.9, L4.5, T7.4	$\lneq$, L4.5, T7.4
$T(3,0,1)$	$\lneq$, L4.4, T8.2	$\lneq$, L4.4, T7.4	$\lneq$, L4.4, T7.4
$T(3,1,1)$	$\lneq$, L4.4, T8.2	$\lneq$, L4.5, T7.4	$\lneq$, L4.4, T7.4
$T(3,2,1)$		$\not>$, T7.4	$\lneq$, L4.5, T7.4
$T(3,0,2)$			$\lneq$, L4.4, T8.2

We display the order diagram of $\mathbb{T}$ in Figure 19. In this diagram, we use a solid edge to represent a known comparability, a missing edge to represent a known incomparability, and a dotted edge to represent

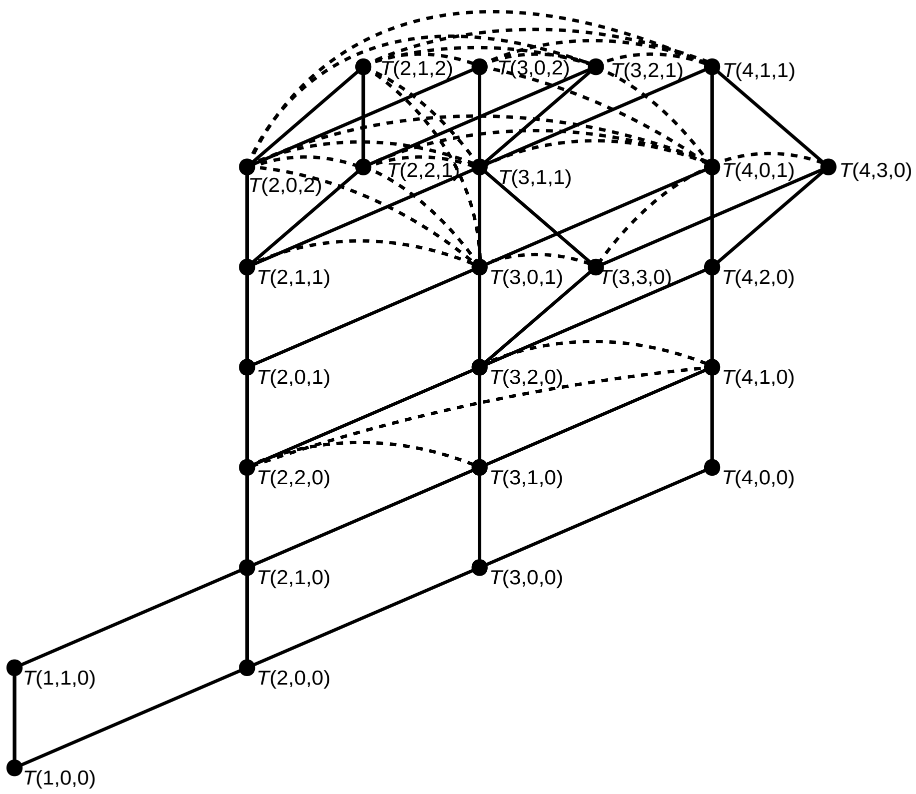


Figure 19. The classes of tube orders for small values of n, i, and f

an unknown relation. There are clearly an infinite number of classes of tube orders; we include only the first few values of n, i, and f in the diagram.

The dotted edges occur more frequently for larger values of n, i, and f. Of course, we wish to know the relations that are currently unknown. More significantly, we would like to discover a pattern in the relations that might lend more insight into general (n, i, f)-tube orders. The most recently discovered relations between classes of (n, i, f)-tube orders follow no clear pattern.

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