A BORSUK-ULAM EQUIVALENT THAT DIRECTLY IMPLIES SPERNER'S LEMMA

KATHRYN L. NYMAN AND FRANCIS EDWARD SU

Abstract. In this note, we show that Fan's 1952 lemma on labelled triangulations of the n-sphere with $n+1$ labels is equivalent to the Borsuk-Ulam theorem. Moreover, unlike other Borsuk-Ulam equivalents, this lemma directly implies Sperner's Lemma, so this proof may be regarded as a combinatorial version of the fact that the Borsuk-Ulam theorem implies the Brouwer fixed point theorem, or that the Lusternik-Schnirelmann-Borsuk theorem implies the KKM lemma.

1. Introduction

The Brouwer fixed point theorem, the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma, and Sperner's lemma are known to be equivalent. Equally powerful, they form a triumvirate of theorems whose interconnections have been exploited with great success in fixed point algorithms [12, 14] as well as in game theory [1]. Similarly, the Borsuk-Ulam theorem, the Lusternik-Schnirelmann-Borsuk (LSB) theorem, and Tucker's lemma are another triumvirate of equivalent results. In each of these triples, the first is a topological result, the second is a set-covering result, and the third is a combinatorial result.

Moreover, these triples are related to each other. Since the Borsuk-Ulam theorem implies the Brouwer fixed point theorem, any theorem in the second triple must imply any theorem in the first. It is an interesting question to find *direct* proofs of each implication. For instance, a topological construction shows how a Brouwer fixed point follows from Borsuk-Ulam antipodes [11], and with set-coverings the LSB theorem can be used to directly prove the KKM lemma [9]. But in the combinatorial domain, we are unaware of a direct proof that Tucker's lemma implies Sperner's lemma.

In this article, we show that another combinatorial lemma, Fan's N+1 Lemma, may be a more natural combinatorial analogue of the Borsuk-Ulam theorem, and therefore more worthy to sit in the Borsuk-Ulam triumvirate than Tucker's lemma. In particular, in Section 3 we show that Fan's N+1 Lemma is equivalent to the Borsuk-Ulam theorem, and in Section 4 we exhibit a direct proof that it implies Sperner's lemma.

2. Background

We first review these theorems. Let Σ^n be a combinatorial version of the n-sphere, the set of all points in $\mathbb{R}^{n+1}$ of length 1 in the L_1 norm:

$$\Sigma^n = \{(x_1, ..., x_{n+1}) : \sum |x_i| = 1\}.$$

In $\mathbb{R}^3$, Σ^2 is just the boundary of the octahedron. As with the octahedron, note that Σ^n is naturally subdivided into orthants; we will study labelled triangulations of Σ^n that refine the orthant subdivision. A *triangulation* is a subdivision by simplices that either meet face-to-face or not at all. Each simplex is the affine hull of its *vertices*; these are the *vertices of the triangulation*. A triangulation of Σ^n is *symmetric* if when σ is a simplex of the triangulation, then $-\sigma$ is a simplex as well.

Define an *m-labelling* to be a function ℓ that assigns to each vertex v one of $2m$ possible integers: $\{\pm1, \pm2, \dots, \pm m\}$. A symmetric triangulation of Σ^n has an *anti-symmetric* labelling if $\ell(-v) = -\ell(v)$ for all vertices v. A labelling has a *complementary edge* if some adjacent pair of vertices has labels that sum to zero, e.g., $\{+i, -i\}$.

Call a simplex *alternating* if its vertex labels are distinct in magnitude and alternate signs, when arranged in order of increasing value. So the labels have the form

$$\{k_1, -k_2, k_3, \dots\} \qquad \text{or} \qquad \{-k_1, k_2, -k_3, \dots\}$$

Date: January 12, 2012.

This work was supported in part by NSF Grants DMS-0701308 and DMS-1002938.

when $1 \leq k_1 < k_2 < k_3 < \cdots$. The first kind is called *positive alternating* and the second is *negative alternating*, based on the sign of k_1. For instance a triangle labelled $\{-1, +3, -7\}$ would be negative alternating, and an edge labelled $\{+2, -3\}$ would be positive alternating.

Fan's N+1 Lemma. *Let T be a symmetric triangulation of Σ^n with an $(n+1)$-labelling that is anti-symmetric and has no complementary edge. Then T has a positive alternating n-simplex.*

Thus if the boundary of an octahedron (e.g, see Figure 3) has a triangulation anti-symmetrically labelled by $\{\pm 1, \pm 2, \pm 3\}$ and no complementary edges, then it must have a $\{+1, -2, +3\}$ triangle.

We call this Fan's N+1 Lemma because Fan's original lemma [4] is more general: it says that for any m-labelling with the same hypotheses, there are an odd number of positive alternating n-simplices and an equal number of negative alternating n-simplices. And as [8] shows, the result holds for more general triangulations of S^n with a constructive proof. When $m = n+1$, an m-labelling has only one kind of positive alternating simplex— namely, the simplex with labels of every magnitude: $\{1, -2, +3, \ldots, (-1)^n(n+1)\}$.

Note that if an anti-symmetric m-labelling has no complementary edge, then $m \geq n+1$, because alternating simplices must have $n+1$ different label values (apart from sign). Since an n-labelling is an $(n+1)$-labelling with one label missing, then as noted by Fan [4], the contrapositive of Fan's N+1 Lemma yields Tucker's lemma as a corollary:

Tucker's Lemma. *Let T be a symmetric triangulation of Σ^n with an n-labelling that is anti-symmetric. Then T has a complementary edge.*

Tucker's lemma[6, 13] was originally proposed as a combinatorial equivalent of the Borsuk-Ulam theorem [2]:

Borsuk-Ulam Theorem. *Let $h : S^n \to \mathbb{R}^n$ be a continuous function such that $h(-x) = -h(x)$ for all $x \in S^n$. Then there exists $w \in S^n$ such that $h(w) = 0$.*

A set covering result due to Lusternik-Schnirelman-Borsuk[2, 7] is also equivalent to the Borsuk-Ulam theorem:

LSB Theorem. *Let $C_1, ..., C_{n+1}$ be a collection of closed sets that cover S^n. Then at least one of the sets must contain a pair of antipodal points.*

These theorems (Fan, Tucker, Borsuk-Ulam, LSB) concern topological or combinatorial n-spheres. The next three theorems concern topological and combinatorial n-balls.

Let B^n denote an n-ball, the set of all points within unit distance of the origin in $\mathbb{R}^n$. A combinatorial version of a n-ball is a n-simplex, which is more naturally described by its embedding in $\mathbb{R}^{n+1}$:

$$\Delta^n = \{(x_1, ..., x_{n+1}) : x_i \geq 0, \sum x_i = 1\}.$$

It is homeomorphic to an n-ball. For any $v = (v_1, ..., v_{n+1}) \in \Delta^n$, let

$$Z(v) = \{i : v_i \neq 0\}$$

be the set of indices of coordinates of v that are non-zero. Thus in Δ^2, $Z((0,1,0)) = \{2\}$ and $Z((.3, 0, .7)) = \{1, 3\}$. Suppose T is a triangulation of Δ^n. A *Sperner-labelling* ℓ assigns to each vertex v a label from $\{1, ..., n+1\}$ such that

$$\ell(v) \in Z(v). \tag{1}$$

This forces each main vertex of Δ^n to have a different label (the index of its one non-zero coordinate), and any vertex on a face of Δ^n can only be labelled by one of the main vertices that span that face. Call an n-simplex in the triangulation *fully-labelled* if its vertices have distinct labels (and therefore all labels $\{1, ..., n+1\}$).

Sperner's Lemma. *Any Sperner labelled triangulation of Δ^n must have a fully-labelled n-simplex.*

In fact, there are an odd number of such simplices [10]. Sperner's lemma provides the simplest route to proving this famous theorem of Brouwer [3]:

Brouwer Fixed Point Theorem. *For any continuous function $f : B^n \to S^n$, there exists a point $x \in S^n$ such that $f(x) = x$.*

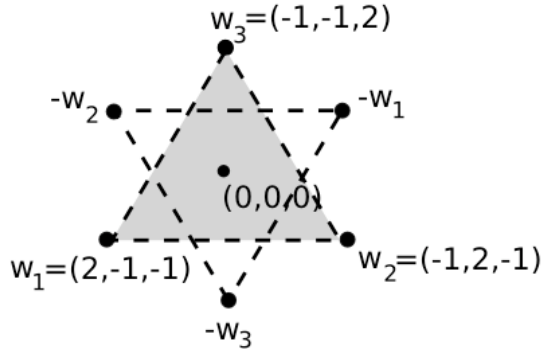

FIGURE 1. For $n = 2$, the points w_1, w_2, w_3 and $-w_1, -w_2, -w_3$ in the hyperplane H. The shaded region is the image under h of a positive alternating 2-simplex, which contains all the positive w_i (and the origin).

Knaster-Kuratowski-Mazurkiewicz [5] provided the original link between the Brouwer theorem and Sperner's lemma:

KKM Lemma. *Let $C_1, ..., C_n$ be a collection of closed sets that cover Δ^n such that for each $I \subseteq [n+1]$, the face spanned by the set $\{e_i | i \in I\}$ is covered by $\{C_i | i \in I\}$. Then $\cap_{i=1}^n C_i$ is non-empty.*

3. Equivalence of Fan's N+1 Lemma and the Borsuk-Ulam Theorem

As discussed earlier, it has been known that Fan's general lemma with m-labellings [4] implies the Borsuk-Ulam Theorem through Tucker's lemma. Here we show that Fan's N+1 Lemma is equivalent to the Borsuk-Ulam theorem.

Theorem 1. *Fan's N+1 Lemma is equivalent to the Borsuk-Ulam Theorem.*

Proof. We first show the Borsuk-Ulam Theorem implies Fan's N+1 Lemma. Let T be a symmetric triangulation of Σ^n with an anti-symmetric $(n+1)$-labelling L in which there are no complementary edges. Let $w_i \in \mathbb{R}^{n+1}$ be the point with ith coordinate n and other coordinates -1:

$$w_i = (-1, \ldots, -1, n, -1, \ldots - 1).$$

Let $W_+ = \{w_1, ..., w_{n+1}\}$ and $W_- = \{-w_1, ..., -w_{n+1}\}$. The set $W = W_+ \cup W_-$ comprises $2n+2$ points that lie on the n-dimensional hyperplane: $H = \{(x_1, \ldots, x_{n+1}) : \sum_{i=1}^{n+1} x_i = 0\}$.

Define a continuous map $h : \Sigma^n \to H$ as follows. For each $v \in T$, let

$$h(v) = \begin{cases} w_{L(v)} & \text{if } L(v) \text{ is odd} \\ -w_{L(v)} & \text{if } L(v) \text{ is even,} \end{cases} \tag{2}$$

where $w_{-i} = -w_i$ in case $L(v) < 0$. Extend h linearly to each simplex of T. Since L is an anti-symmetric labelling, we see $h(-x) = -h(x)$ for all $x \in \Sigma^n$. Therefore, by Borsuk-Ulam there is a $z \in \Sigma^n$ such that $h(z) = 0$.

Thus z is in some n-simplex σ such that $h(\sigma)$ contains the origin. The images of the vertices of σ form a set $K = \{h(v) : v \in \sigma, v \in T\}$, a subset of W of size $n+1$ or smaller (if there are repeated labels). Since there are no complementary edges in T, the set K contains no pair $\{w_j, -w_j\}$. Then $K = \{w_j\}_{j \in B} \cup \{-w_j\}_{j \in B'}$ where B and B' are disjoint subsets of $\{1, ..., n+1\}$.

Now consider the sum of vectors in K:

$$v = \sum_{j \in B} w_j - \sum_{j \in B'} w_j.$$

Note that the dot products $w_i \cdot w_i = n(n+1)$ for all $i \in [n+1]$, and $w_i \cdot w_j = -(n+1)$ for all $j \neq i$. So, for $i \in B$, the dot product

$$w_i \cdot v = n(n+1) - (|B| - 1)(n+1) + |B'|(n+1) = (n+1)(n+1 - |B| + |B'|),$$

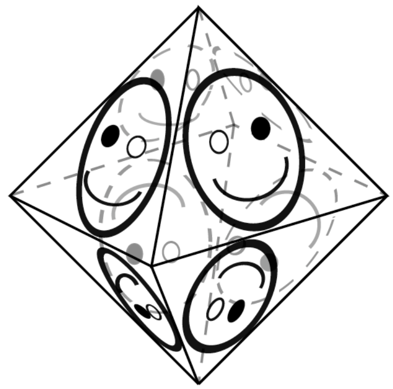

FIGURE 2. The action of G on Σ^n, as shown by their effect on Mr. Smiley.

which is positive unless $|B| = n+1$ and $|B'| = 0$, i.e., $K = W_+$. And for $i \in B'$,

$$-w_i \cdot v = |B|(n+1) + n(n+1) - (|B'|-1)(n+1) = (n+1)(|B| - |B'| + n + 1),$$

which is positive unless $|B'| = n+1$ and $|B| = 0$, i.e., $K = W_-$. Since the convex hull of K contains the origin, it cannot be the case that all vectors in K have a positive dot product with v. So either $K = W_+$ or $K = W_-$ (and indeed, in these cases, K's convex hull contains the origin).

If $K = W_+$, then (2) shows the original simplex σ has labels $\{1, -2, \ldots, (-1)^n n+1\}$. If $K = W_-$, then (2) and anti-symmetry of L shows that $-\sigma$ has these labels. In either case we find a positive alternating simplex, as desired.

Now we show Fan's N+1 Lemma implies the Borsuk-Ulam Theorem. Let $h : \Sigma^n \to \mathbb{R}^n$ be a continuous function such that $h(-x) = -h(x)$ for all $x \in \Sigma^n$. Assume, by way of contradiction, that there is no point $z \in \Sigma^n$ such that $h(z) = 0$. If $h(x) = (x'_1, \ldots, x'_n)$, let $\hat{h} : \Sigma^n \to \mathbb{R}^{n+1}$ be the function defined by $\hat{h}(x) = (x'_1, \ldots, x'_n, -\sum_{i=1}^n x'_i)$. So $\hat{h}$ maps Σ^n to the hyperplane H and preserves continuity and anti-symmetry. Furthermore, there is no point z such that $\hat{h}(z) = 0$.

Let T be a symmetric triangulation of Σ^n, and let the set W be as above. We wish to construct a labelling L on the vertices of T that is anti-symmetric.

Define $L(v)$ to be the index i such that w_i is closest to $\hat{h}(v)$ in $\mathbb{R}^{n+1}$ for all $i \in \{\pm 1, ..., \pm(n+1)\}$. In the case of ties, choose the index with the smallest absolute value. This is well-defined because $\hat{h}(v)$ is never 0, and no non-zero point can be equidistant from w_i and $w_{-i} = -w_i$. That L is anti-symmetric follows from noting that $\hat{h}$ is anti-symmetric, so $\hat{h}(v)$ is closest to w_i if and only if $\hat{h}(-v)$ is closest to w_{-i}.

Therefore, by Fan's N+1 Lemma, there exists either a complementary edge $(+i, -i)$, for some i, or an alternating simplex with labels $\{1, -2, \ldots, (-1)^n n+1\}$. By taking finer and finer triangulations, and by the compactness of the Σ^n, there exists a convergent subsequence of shrinking positive alternating simplices or a convergent subsequence of shorter complementary edges involving the same index i. This gives a limit point which, by the continuity of $\hat{h}$, is either equidistant from both w_i and $-w_i$, or is equidistant from all points in $\{w_1, -w_2, w_3, ..., (-1)^n w_{n+1}\}$. But the only point with this property is 0. Thus, the limit point z must satisfy $\hat{h}(z) = 0$ and therefore, $h(z) = 0$.

□

4. Fan's N+1 Lemma implies Sperner's lemma

Now we establish how Fan's N+1 Lemma will indeed prove Sperner's Lemma by a direct construction, so it is the "right" combinatorial result to sit in the Borsuk-Ulam triumvirate.

Theorem 2. *Fan's N+1 Lemma implies Sperner's lemma.*

Proof. Consider a triangulation S of Δ^n with a Sperner-labelling ℓ. We first extend S to a triangulation T of Σ^n by reflecting copies of S to the other orthants of Σ^n. Let $G = \{\pm 1\}^{n+1}$ denote the group of symmetries of

Σ^n generated by reflections that flip the sign of selected coordinates; then the action of $g = (g_1, ..., g_{n+1}) \in G$ on $v = (v_1, ..., v_{n+1}) \in \Sigma^n$ produces $gv = (g_1 v_1, ..., g_{n+1} v_{n+1}) \in \Sigma^n$. So g reflects v in all coordinates i for which $g_i = -1$. Note $g = (1, 1, ..., 1)$ is the identity in G. The idea of this construction is illustrated in Figure 2.

Similarly if σ is a simplex in S spanned by a set of vertices V, we define $g\sigma$ to be the simplex spanned by the vertices in $gV = \{gv : v \in V\}$. Let T be the collection of simplices $\{g\sigma : \sigma \in S \text{ and } g \in G\}$. This results in a triangulation of Σ^n since the reflection method ensures that simplices of T meet face-to-face along reflected facets of S.

Now we extend the labelling ℓ on vertices of S to a labelling L on vertices of T by reflection but with possible sign modifications. Define:

$$L(gv) = g_{\ell(v)} \cdot (-1)^{\ell(v)+1} \cdot \ell(v) \tag{3}$$

for each $v \in S$. Notice that $L(gv)$ and $\ell(v)$ have the same label value (but possibly different signs). When $g = (1, 1, ..., 1)$, this defines L on S and the factor $(-1)^{\ell(v)+1}$ turns fully-labelled simplices into positive alternating simplices. When g is non-trivial, L defines a labelling of vertices on reflected copies of S (see Figure 3).

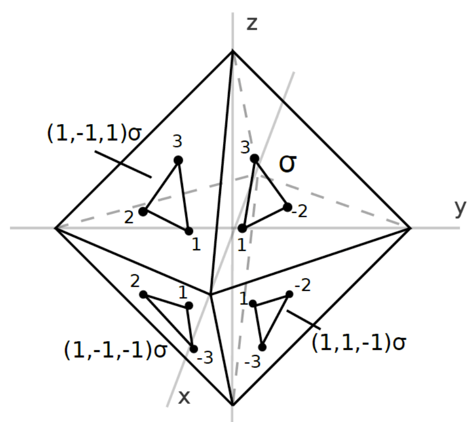

FIGURE 3. A positive alternating simplex σ in T arising from a fully-labelled simplex with labels $\{1, 2, 3\}$ in S, and reflected simplices $g\sigma$ for $g = (1, 1, -1), (1, -1, 1)$ and $(1, -1, -1)$ with their L-labellings indicated.

We might worry that L is not well-defined where orthants meet. However, orthants meet where $gv = \hat{g}\hat{v}$ for some $g, \hat{g} \in G$ and some $v, \hat{v} \in S$. But then $g_i v_i = \hat{g}_i \hat{v}_i$ for each i, which implies $v_i = \hat{v}_i$ since $g_i, \hat{g}_i = \pm 1$. Then $g_i = \hat{g}_i$ when $v_i \neq 0$, i.e., when $i \in Z(v)$. But $\ell(v) \in Z(v)$ by (1), so that $g_{\ell(v)} = \hat{g}_{\ell(v)}$. It follows from (3) that $L(gv) = L(\hat{g}v)$, so L is well-defined.

Now we show that L satisfies the conditions of Fan's N+1 Lemma. Antipodal labels sum to zero by construction: the point antipodal to v is $-v = \bar{g}v$, where $\bar{g} = (-1, -1, \ldots, -1)$, so that (3) gives $L(-v) = -L(v)$. Also, we can show L has no complementary edges. Every edge in T is a reflected copy of some edge in S via some $g \in G$, and the Sperner labelling ℓ of S has no complementary edges (all labels are positive). Then the rule (3) shows that for any choice of g, two vertices $v, w \in S$ will have identical ℓ-labels ($\ell(v) = \ell(w)$) if and only if their g-reflections have identical L-labels as well ($L(gv) = L(gw)$). So L has no complementary edges, because ℓ did not.

Thus Fan's N+1 Lemma applies so there exists a positive alternating n-simplex in T. Since Δ^n is the only facet of Σ^n that contains the labels $\{1, -2, 3, \ldots, (-1)^n n + 1\}$, there must be a fully-labeled n-simplex in S. □

In fact, as noted earlier, a stronger version of Fan's N+1 Lemma holds whose conclusion is that there are in fact an odd number of positive alternating n-simplices. Then the above argument would demonstrate the stronger version of Sperner's lemma that concludes there are an odd number of fully-labelled n-simplices in S.

References

1. K. C. Border, *Fixed point theorems with applications to economics and game theory*, Cambridge Univ. Press, Cambridge, 1985.
2. K. Borsuk, Drei Sätze über die n-dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933) 177–190.
3. L. E. J. Brouwer, Über Abbildungen von Mannigfaltigkeiten, *Math. Ann.*, **71** (1912) 97–115.
4. K. Fan. A generalization of Tucker's combinatorial lemma with topological applications. *Ann. of Math.* (2), **56** (1952), pp. 431–437.
5. B. Knaster, C. Kuratowski, and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, *Fund. Math.* **14** (1929) 132–137.
6. S. Lefshetz, *Introduction to Topology*, pp. 134-141. Princeton University Press, Princeton, NJ, 1949.
7. L. Lusternik and L. Schnirelman, *Topological Methods in Variational Calculus*, Issledowatelskiĭ Institut Matematiki i Mechaniki pri O.M.G.U., Moscow, 1930 [Russian].
8. T. Prescott and F. Su, A constructive proof of Ky Fan's generalization of Tucker's lemma, *J. Combin. Theory Ser. A* **111** (2005), pp. 257–265.
9. G. Spencer and F. E. Su. The LSB theorem implies the KKM lemma. *Amer. Math. Monthly* **114** (2007), pp. 156–159.
10. E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg* **6** (1928), pp. 265–272.
11. F. E. Su. Borsuk-Ulam implies Brouwer: a direct construction. *Amer. Math. Monthly* **104** (1997), pp. 855–859.
12. Todd, M.J., 1976. *The Computation of Fixed Points and Applications*. Lecture Notes in Economic and Mathematical Systems, Springer-Verlag, New York.
13. A. W. Tucker, Some Topological Properties of the Disk and Sphere, *Proc. First Canad. Math. Congr.*, Montreal, 285–309, 1945.
14. Yang, Z., 1999. *Computing Equilibria and Fixed Points*. Kluwer Academic Publishers, Boston.

Department of Mathematics, Willamette University
E-mail address: knyman@willamette.edu

Department of Mathematics, Harvey Mudd College
E-mail address: su@math.hmc.edu