Integration by Parts review and shortcut
Math 256 Fall, 2012

This handout will review integration by parts, particularly repeated integration by parts, giving a substantial shortcut for the latter.

Integration by Parts:

Integration by parts is the integration counterpart to the product rule, but it is not *quite* the reverse of the product rule. The method, in short, is this:

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, f'(x)\, dx.$$

The trick in applying this method is to recognize an integrand (the expression being integrated) as a product of $f(x)$ and $g'(x)$. We usually abbreviate the notation by defining

$$\begin{aligned} u &= f(x) \quad \text{so that } du = f'(x)\, dx \text{ and} \\ v &= g(x) \quad \text{so that } dv = g'(x)\, dx, \end{aligned}$$

so that our rule becomes

$$\int u dv = uv - \int v du.$$

Example:

We want to find $\int xe^{2x} dx$. As a general rule, we want to let u be any power of x that appears in our integrand, if there is one, and we want dv to be something that integrates relatively easily. The latter requirement sometimes overrules the first. This example is straightforward, though: Let $u = x$, then dv is what's left, including the dx. So $dv = e^{2x} dx$. Then, differentiating, $du = dx$, and integrating, $v = \frac{1}{2} e^{2x}$. Then we have

$$\begin{aligned} \int xe^{2x} dx &= \int u \cdot dv \\ &= uv - \int v \cdot du \\ &= (x)\left(\tfrac{1}{2} e^{2x}\right) - \int \left(\tfrac{1}{2} e^{2x}\right)(dx) \\ &= \tfrac{x}{2} e^{2x} - \int \tfrac{1}{2} e^{2x} dx, \end{aligned}$$

with the last integral being easy to finish. Finishing the process, we get

$$\int xe^{2x} dx = \tfrac{x}{2} e^{2x} - \tfrac{1}{4} e^{2x} + C.$$

Second example:

We want to find $\int x^2 e^{3x} dx$. Following the principle in the prior example, let $u = x^2$ and $dv = e^{3x} dx$. Then $du = 2x dx$ and $v = \frac{1}{3} e^{3x}$, and we have

$$\begin{aligned} \int x^2 e^{3x} dx &= \int u \cdot dv \\ &= uv - \int v \cdot du \\ &= \left(x^2\right)\left(\tfrac{1}{3} e^{3x}\right) - \int \left(\tfrac{1}{3} e^{3x}\right)(2x dx) \\ &= \tfrac{1}{3} x^2 e^{3x} - \int \tfrac{2}{3} x e^{3x} dx. \end{aligned}$$

It is important to note both that the last integral is not trivial, but it is easier than the one we started with. We simply repeat the process, this time setting $u = \frac{2}{3} x$ and $dv = e^{3x} dx$. (Note we could assign the $\frac{2}{3}$ to either u or dv, or put it outside the integral altogether.) We then get $du = \frac{2}{3} dx$ and $v = \frac{1}{3} e^{3x}$ and, continuing from the last line above,

$$\begin{aligned} \tfrac{1}{3} x^2 e^{3x} - \int \tfrac{2}{3} x e^{3x} dx &= \tfrac{1}{3} x^2 e^{3x} - \int u \cdot dv \\ &= \tfrac{1}{3} x^2 e^{3x} - \left[uv - \int v \cdot du\right] \\ &= \tfrac{1}{3} x^2 e^{3x} - \left[\left(\tfrac{2}{3} x\right)\left(\tfrac{1}{3} e^{3x}\right) - \int \left(\tfrac{1}{3} e^{3x}\right)\left(\tfrac{2}{3} dx\right)\right] \\ &= \tfrac{1}{3} x^2 e^{3x} - \left(\tfrac{2}{3} x\right)\left(\tfrac{1}{3} e^{3x}\right) + \int \tfrac{2}{9} e^{3x} dx, \end{aligned}$$

with the last integral (finally) being easy. Note carefully the square brackets above and how they require distributing a minus sign to get to the last line. Finishing the process, we get

$$\int x^2 e^{3x} dx = \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{9} x e^{3x} + \tfrac{2}{27} e^{3x} + C.$$

Special Cases to Watch For:

- $\int x^n \ln(x)\, dx$: Let $u = \ln(x)$ and $dv = x^n dx$. After one integration-by-parts application, simplify the new integral algebraically, it will be just a power of x.
- $\int e^{ax} \cos(bx)\, dx$: Let $u = e^{ax}$ and $dv = \cos(bx)$, integrate by parts once. Again let $u = e^{ax}$ in the new integral, and repeat. At first you will feel frustrated in that your second new integral is the one you started with:

$$\begin{aligned} \int e^{ax} \cos(bx)\, dx &= \frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax} \cos(bx) - \int \frac{a^2}{b^2} e^{ax} \cos(bx)\, dx \\ &= \frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax} \cos(bx) - \frac{a^2}{b^2} \int e^{ax} \cos(bx)\, dx. \end{aligned}$$

  However, naming the integral we are seeking as I, we have

$$I = \frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax} \cos(bx) - \frac{a^2}{b^2} I.$$

  Solving for I gives

$$\begin{aligned} I &= \frac{b^2}{a^2 + b^2} \left(\frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax} \cos(bx)\right) + C \\ &= \frac{b}{a^2 + b^2} e^{ax} \sin(bx) + \frac{a}{a^2 + b^2} e^{ax} \cos(bx) + C, \end{aligned}$$

  where the $+C$ comes from our understanding of integration, not directly from the algebra.

- BEWARE: In repeated integration by parts, swapping which part of the integrand is u and which is dv at the second integration will take you in circles, giving a result like

$$\int x^2 e^{3x} dx = \tfrac{1}{3} x^2 e^{3x} - \tfrac{1}{3} x^2 e^{3x} + \int x^2 e^{3x} dx,$$

and attempting to solve for the integral won't help - but what often happens is some error creeps in so that one "can" solve for the integral and get a wrong answer.

A shortcut for repeated parts

First, a general view of what repeated integration by parts looks like. For notational convenience, we need a simpler symbol for the antiderivative of g, so we use $\underline{g}$ for the first antiderivative, and $\underline{\underline{g}}$ for the second antiderivative (i.e. the antiderivative *of* the antiderivative), and so on. Then $\underline{g'(x)} = g(x)$, and we have

$$\int f(x)\, g'(x)\, dx = \int u \cdot dv$$

with $u = f(x)$ and $dv = g'(x)\, dx$ as usual. Continuing,

$$\begin{aligned} &= u \cdot v - \int v \cdot du \\ &= f(x)\, g(x) - \int f'(x)\, g(x)\, dx. \end{aligned}$$

Now taking $u = f'(x)$ and $dv = g(x)\, dx$, we get $du = f''(x)\, dx$ and $v = \underline{g(x)}$, so continuing,

$$\begin{aligned} &= f(x)\, g(x) - \left[uv - \int v \cdot du \right] \\ &= f(x)\, g(x) - f'(x)\, \underline{g(x)} + \int f''(x)\, \underline{g(x)} dx. \end{aligned}$$

Continuing in this vein, always differentiating the f and anti-differentiating the g, we get

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - f'(x)\, \underline{g(x)} + f''(x)\, \underline{\underline{g(x)}} - f'''(x)\, \underline{\underline{\underline{g(x)}}} + f''''\underline{\underline{\underline{\underline{g(x)}}}} - \dots$$

...which can be continued indefinitely, but the hope is that eventually the higher derivatives of f will be zero, and the sum will end.

Repeated integration by parts is easily done using a table, for example, for the integral $\int x^2 e^{3x} dx$, we choose $f(x) = x^2$ and $g'(x) = e^{3x}$, then make a table:

f	g'	$+/-$
x^2	e^{3x}	$+$
$2x$	$\frac{1}{3}e^{3x}$	$-$
2	$\frac{1}{9}e^{3x}$	$+$
0	$\frac{1}{27}e^{3x}$	$-$
0	$\frac{1}{81}e^{3x}$	$+$
0	$\frac{1}{243}e^{3x}$	$-$

The column labeled "f" lists f and its successive derivatives, the column labeled "g'" lists g' and its successive antiderivatives, and the third column alternates sign, beginning with "+". Notice how our earlier example, $\int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C$, can be read by simply multiplying on the downward diagonals and adding:

$$(x^2)\left(\frac{1}{3}e^{3x}\right)(+) + (2x)\left(\frac{1}{9}e^{3x}\right)(-) + (2)\left(\frac{1}{27}e^{3x}\right)(+) + (0)\left(\frac{1}{81}e^{3x}\right)(-) + \cdots$$

After the first three terms subsequent products are zero and thus we can stop.